Axisymmetry in Numerical Relativity

Introduction

Why axisymmetry?

1. Simpler than 3D (but problems with coordinate singularities!)

2. Astrophysically relevant—rotation.

Highly selective and incomplete literature survey:

- Kaluza-Klein reduction: R. Geroch, *J. Math. Phys.*, **12**, 918(1971). Vacuum only

- Numerical schemes

1. T. Nakamura et al., *Prog. Theor. Phys. Suppl.*,**90**, 1 (1987) based on work 1980–82, includes rotation, hydro. NOK.
2. D. Garfinkle & G.C. Duncan, *Phys. Rev.*, **D63**, 044011 (2000) no rotation, vacuum. GD
3. M. Choptuik et al., *Class. & Quant. Grav.*, **20**, 1857 (2003) no rotation, vacuum or scalar field. CHLP.
4. A.P. Barnes, Ph.D. thesis, Cambridge (2004) rotation, stars. B.
5. O. Rinne & J.M. Stewart, *Class. & Quant. Grav.*, **22**, 1143–1166 (2005) rotation, general matter. RS.
6. O. Rinne, Ph.D. thesis, Cambridge (2005) rotation, vacuum. R.

Ingredient 1: Cylindrical Polar coordinates

Local Minkowski coordinates $X^\mu = (t, z, x, y)$, $x^\mu = (t, z, r, \varphi)$. There is a coordinate singularity at $r = 0$, e.g., look at Killing vector

$$\xi = \begin{cases} \partial/\partial\varphi & \text{provided } r \neq 0, \\ y\partial/\partial x - x\partial/\partial y & \text{everywhere.} \end{cases}$$

A geometric quantity Q is **axisymmetric and regular on axis** if

1. $\mathcal{L}_\xi Q = 0$,

2. the (t, z, x, y) components of Q admit (possibly truncated) Taylor series expansions wrt x and y in a neighbourhood of $x = y = 0$.

For a scalar f this means $f = f(t, z, r^2)$.

A symmetric tensor field $p_{\alpha\beta}$ is both axisymmetric and regular on axis iff its (t, z, r, φ) components satisfy

$$p_{\alpha\beta} = \begin{pmatrix} A & B & rD & r^2F \\ B & C & rE & r^2G \\ rD & rE & H + r^2J & r^3K \\ r^2F & r^2G & r^3K & r^2(H - r^2J) \end{pmatrix}.$$

Here $A, B, \ldots$ are functions of t, z and r^2. This result is well known, but does not appear to be in the literature.

Ingredient 2: Kaluza-Klein Reduction

The orbits of ξ^μ form a 3-dim. manifold $\mathcal{N}$. We can project tensors $Q^{\alpha\cdots}{}_{\beta\ldots}$ from $\mathcal{M}$ to $\mathcal{N}$ provided

1. $\mathcal{L}_\xi Q = 0$,

2. $Q^{\alpha\cdots}{}_{\beta\ldots}\xi^\beta = 0$ etc.

I. What is the reduction of Einstein's theory from $\mathcal{M}$ to $\mathcal{N}$?

II. If the geometry of $\mathcal{N}$ is determined how can that of $\mathcal{M}$ be reconstructed?

The reduction $\mathcal{M} \to \mathcal{N}$ for vacuum is due to Geroch. **NOK** included a perfect fluid and **BRS** included general matter. **CHLP** is similar but no rotation.

Geroch described the transition $\mathcal{N} \to \mathcal{M}$. Difficult to implement numerically but, fortunately, sensible physical questions in $\mathcal{M}$ seem to have answers in $\mathcal{N}$. E.g., gravitational radiation. The leading term in Ψ_4 measured wrt asymptotic NP tetrad in $\mathcal{M}$ can be evaluated from quantities determined in $\mathcal{N}$.

$\mathcal{N}$ has signature $(+--)$ and so we can perform an "ADM" reduction, *the (2+1)+1 approach*.

One novelty is that the rotation variables in $\mathcal{N}$, the components of the twist vector (curl of the Killing vector projected into $\mathcal{N}$), satisfy equations whose principal part is that of (axisymmetric) Maxwell equations. They couple to matter and geometry only through source terms.

Ingredient 3: Regular Variables & Equations

The first ingredient tells us the behaviour of all quantities near the axis $r = 0$. We redefine the dependent variables $\{Q\}$ to be of the form

$$Q = f(t, z, r^2) \text{ or } Q = rg(t, z, r^2),$$

so that we can impose either Neumann ($\partial Q/\partial r = 0$) or Dirichlet ($Q = 0$) boundary conditions at $r = 0$. Thus our dependent variables are manifestly regular on axis $r = 0$.

We can also redefine them[1], maintaining regularity, so that the new equations are manifestly regular at the axis (no $r^{-1}, r^{-2}, \ldots$ factors) provided the dependent variables satisfy the boundary conditions above.

[1]You need a competent computer algebra package, e.g., REDUCE, to verify this.

Flavouring: Gauge Choices

We have almost all of the ingredients to make a numerical algorithm. But we need to make gauge and other choices. Since these are a matter of taste we call them flavourings.

The first class of flavours comes from noting that the spatial metric H_{AB} is 2D and all 2D metrics are conformally flat so that we can choose the spatial coordinates to set

$$H_{AB} = \begin{pmatrix} \psi^4 & 0 \\ 0 & \psi^4 \end{pmatrix}.$$

There are at least three subclasses, each favoured by different groups.

Free evolution used by **GD** solves the minimum number of elliptic equations. The shift vector β^A is determined as the solution of gauge conditions implied by the choice of H_{AB} above and its time derivative. Maximal slicing generates an elliptic equation to determine the lapse α. Everything else is determined by a hyperbolic evolution system.

Although this looks very plausible we found it rather difficult to implement because of instabilities. If we denote the constraints by $\mathbf{C}$ we can derive an evolution equation for them

$$\partial_t \mathbf{C} = F^A \partial_A \mathbf{C} + G\mathbf{C},$$

so that if $\mathbf{C} = 0$ initially then $\mathbf{C} \equiv 0$. However the matrix F^r has complex eigenvalues and so this is not a hyperbolic system—the IVP for the constraints is ill-posed. We can modify the slicing condition by adding an appropriate multiple of the energy constraint so as to make the above system hyperbolic. Unlike **GD** we have chosen to solve elliptic equations using Multigrid techniques so as to obtain computationally efficient algorithms. However Multigrid fails for the new slicing condition because the underlying matrix becomes indefinite, even for weak perturbations of flat spacetime.

Another subclass is **constrained evolution** favoured by **CHLP**, where during the evolution elliptic constraint equations are solved for α, β^A and ψ. We found problems with the slicing and energy constraint equations. Ignoring twist and matter terms **R** determines the latter to be

$$\Delta\psi + K^2\psi^p = 0 \qquad \text{in } \Omega, \qquad \psi = 1 \quad \text{on } \partial\Omega,$$

where K^2 is the square of the extrinsic curvature and p $=$ 5. This is not linearization stable and there is no maximum principle to imply existence and uniquess. It can be shown[2] that the Dirichlet problem has at least one nontrivial weak solution provided

$$1 < p < \frac{n+2}{n-2},$$

[2]See L.C. Evans: *Partial Differential Equations* §8.5.2, Theorem 3.

where n is the number of space dimensions, and $n = 2$ here. Existence (but not uniqueness) is guaranteed, but for largeish K^2 Multigrid fails for this equation (loss of diagonal dominance in the underlying matrix), as observed by **B**, **R** and **CHLP** for strong Brill waves.

Of course we can change the value of p to a negative one (thus guaranteeing linearization stability) by conformally rescaling K. (This is a well-known technique for setting up initial data.) However **R**'s evolutions quickly become unstable! His choice of K is equivalent to that of the **BSSN** system which is known to have good stability properties, and these are lost on rescaling.

Our final subclass is **partially constrained evolution** favoured by **R**. Here the slicing condition and momentum constraints are enforced but not the energy constraint. (There is an evolution equation for ψ.) This seemed to work allowing evolution of weak and strong Brill waves with and without twist. There is a critical amplitude for the initial data separating evolutions which disperse from those which collapse to a black hole. However his code lacked the resolution needed to make quantitative statements. We need, urgently, an adaptive mesh refinement algorithm for mixed elliptic-hyperbolic systems which meets Brandt's requirement of computational efficiency[3].

As another class of flavourings one can eschew mixed elliptic-hyperbolic systems in favour of a completely hyperbolic system. This has two advantages (i) gain control over well-posedness and outer boundary conditions, (ii) can use (Berger-Oliger) AMR.

[3] This is a nontrivial problem, at least to retain the $O(N)$ efficiency of Multigrid, which is being addressed.

There are many equivalent and inequivalent hyperbolic reductions of the EFE. **RS** used the *Z4 system* of Bona et al[4].

$$R_{\alpha\beta} + 2\nabla_{(\alpha}Z_{\beta)} = \kappa(T_{\alpha\beta} - \tfrac{1}{2}T^{\gamma}{}_{\gamma}g_{\alpha\beta}).$$

We impose $\mathcal{L}_\xi Z_\alpha = 0$. If we set $Z_\alpha = (\theta, Z_A, Z^\varphi)$ then the constraints become evolution equations for Z

$$\mathcal{L}_\xi\theta = C_0 + \cdots\,, \qquad \mathcal{L}_\xi Z_A = C_A + \cdots\,, \qquad \mathcal{L}_\xi Z^\varphi = C_\varphi + \cdots\,,$$

where (C_0, C_A, C_φ) are the energy and momentum constraints and $\cdots$ are terms linear in Z. Clearly the constraints are satisfied if $Z = 0$.

[4] Phys. Rev., **D67**, 104005 (2003).

We next introduce dynamical gauge conditions based on the **generalized harmonic gauge condition** of Bona et al[5]. These give evolution equations for α and β^A. One can choose the parameters to obtain a strongly hyperbolic system, but never a symmetric hyperbolic one.

Alternatively one can retain the evolution equation for α and require zero shift, $\beta^A = 0$. Again the system is strongly hyperbolic, and, in the special case of harmonic gauge, the system is symmetric hyperbolic.

Using a computer algebra package one can redefine the dependent variables so that every dependent variable and every term in every equation of our first order system is manifestly regular on axis. (Note that even in flat spacetime, cylindrical polar coordinates are not harmonic. We need to use the *gauge source functions* of Friedrich[6] to ensure regularity on axis.)

[5] Phys. Rev., **D67**, 104005 (2003).

[6] Class. & Quantum Grav., **13**, 451 (1996).

Indeed we can even write the system in conservation form

$$\partial_t \mathbf{u} + \partial_A \left[-\beta^A \mathbf{u} + \alpha \mathcal{F}^A(\mathbf{u}) \right] = \alpha \mathcal{S}(\mathbf{u}),$$

retaining both these regularity properties and the hyperbolicity attributes.

For many purposes it ius useful to use *characteristic variables*, and here we run into a problem. The transformation conserved $\rightarrow$ characteristic variables is regular on axis, but its inverse in the r-direction is not. This will raise problems if we want to use solution algorithms based on solving Riemann problems. However the discussion of outer boundary conditions is not affected. BCs at $r = r_{max}$ are obviously unaffected. BCs at $z = \pm z_{max}$ are unaffected because the normal direction is the z-direction.

Flavouring: Outer Boundary Conditions

This is a difficult problem and some experience can be gained by looking first at linearized theory.

R looked first at *dissipative BCs*, and considered two such strategies

1. *absorbing BCs*: incoming modes are set to zero. Unfortunately the "exact" solutions of linearized theory do not obey them.

2. *vanishing of* Z_α. These are satisfied to leading order in linearized theory but perform poorly, compared with 1, in numerical experiments.

One can do better than this.

1. Demand *no incoming radiation*, i.e., with a suitable Newman-Penrose tetrad defined at the outer boundaries, $\Psi_0 = 0$. This gives two real conditions which are satisfied to order $O(r^{-5}, z^{-5})$ in linearized theory.

2. Demand *constraint preserving BCs*, i.e., if we look at the subsidiary system governing the evolution of the constraints, all incoming modes are set to zero. Since our linearized theory solutions satisfy the constraints to this order they satisfy this condition.

3. Demand *gauge preserving BCs*, i.e., do the same as item 2 but for the evolution system for α and β^A. Ditto.

This gives nine conditions at each boundary, or seven for zero shift. In axisymmetry this is precisely the number required. The normal derivatives of the incoming modes are specified in terms of the tangential derivatives and source terms. **R** has checked them against linearized theory. Using GKS theory he has checked a necessary condition for the well-posedness of the IBVP in the high frequency limit.

Let me close with some very preliminary simulations carried out by **R**. **R**'s current evolutions are of Brill waves with twist, and the end product of a low amplitude subcritical evolution will be flat spacetime. However the final line element is flat but not Minkowskian! The variable $e^s(t,z,r)$ measures the ratio of circumference-radius to coordinate-radius and is zero in Minkowski spacetime. Note that the $r=0$ axis (left edge) is totally stable and there is minimal reflection from the far-too-close outer boundaries. s

Alcubierre has suggested that a dynamically evolved lapse can produce "gauge shocks", chart discontinuities, which render the subsequent evolution useless. The second movie is a plot of $\alpha_{,r}/\alpha$ as a function of r and z as t increases. The boxes show the extent of the AMR subgrids. Although the function looks "spiky" it isn't! The still shows $\alpha_{,r}/\alpha$ at a fixed time on the finest AMR-generated subgrid. A_r, $A_r$closeup

The third movie shows B^φ evolving. (The twist variables E^r, E^z and B^φ obey Maxwell equations.) B^φ

Conclusions

- Axisymmetric evolutions which remain regular on axis are feasible, and there are no constraints on the algorithms to be used.
- It's essential to be aware of the underlying mathematics, existence, uniqueness, stability etc.
- A competent computer algebra package is very useful.
- AMR is no longer a luxury; consider e.g., the "gauge shock".
- Remember Brandt's dictum: concentrate the effort where the physics is interesting.