Simulations of Binary Black Hole Spacetimes

Frans Pretorius
University of Alberta,
California Institute of Technology

New Directions In Numerical Relativity
August 18-19, 2005
University of Southampton

Outline

- Methodology
  - an evolution scheme based on generalized harmonic coordinates
    - choosing the gauge
    - constraint damping
- Results
  - merger of a close binary
  - an early look at not-so-close binaries
- Summary
  - near future work

Numerical relativity using generalized harmonic coordinates – a brief overview

- Formalism
  - the Einstein equations are re-expressed in terms of generalized harmonic coordinates
    - add *source functions* to the definition of harmonic coordinates to be able to choose arbitrary slicing/gauge conditions
  - add constraint damping terms to aid in the stable evolution of black hole spacetimes

- Numerical method
  - equations discretized using finite difference methods
  - *directly* discretize the metric; i.e. no "conjugate variables" introduced
  - use adaptive mesh refinement (AMR) to adequately resolve all relevant spatial/temporal length scales (still need supercomputers in 3D)
  - use (dynamical) excision to deal with geometric singularities that occur inside of black holes
  - add numerical dissipation to eliminate high-frequency instabilities that otherwise tend to occur near black holes
  - use a coordinate system compactified to spatial infinity to place the physically correct outer boundary conditions

Generalized Harmonic Coordinates

- Generalized harmonic coordinates introduce a set of arbitrary *source functions* H^{μ} into the usual definition of harmonic coordinates

$$\boxed{\nabla^{\alpha}\nabla_{\alpha}x^{\mu} \equiv \frac{1}{\sqrt{-g}}\partial_{\alpha}\left(\sqrt{-g}\,g^{\alpha\mu}\right) = H^{\mu}}$$

- When this condition (specifically its gradient) is substituted for certain terms in the Einstein equations, *and the H^{μ} are promoted to the status of independent functions*, the principle part of the equation for *each* metric element reduces to a simple wave equation

$$\boxed{g^{\gamma\delta}g_{\alpha\beta,\gamma\delta} + \ldots = 0}$$

Generalized Harmonic Coordinates

- The claim then is that a solution to the coupled Einstein-harmonic equations

$$\boxed{\begin{aligned} &g^{\gamma\delta} g_{\alpha\beta,\gamma\delta} + 2 g^{\gamma\delta}{}_{,(\alpha} g_{\beta)\delta,\gamma} + 2 H_{(\alpha,\beta)} - 2 H_{\delta} \Gamma^{\delta}_{\alpha\beta} + 2 \Gamma^{\gamma}_{\delta\beta} \Gamma^{\delta}_{\gamma\alpha} \\ &+ 8\pi \left(2 T_{\alpha\beta} - g_{\alpha\beta} T \right) = 0 \end{aligned}}$$

which include (arbitrary) evolution equations for the source functions, plus additional matter evolution equations, will also be a solution to the Einstein equations *provided* the harmonic constraints

$$\boxed{C^{\mu} \equiv H^{\mu} - \nabla^{\alpha} \nabla_{\alpha} x^{\mu}}$$

and their first time derivative are satisfied at the initial time.

- "Proof" $\boxed{\nabla^{\alpha} \nabla_{\alpha} C^{\mu} = -R^{\mu}{}_{\nu} C^{\mu}}$

An evolution scheme based upon this decomposition

- The idea (following Garfinkle *[PRD 65, 044029 (2002)]*; see also Szilagyi & Winicour *[PRD 68, 041501 (2003)]*) is to construct an evolution scheme based directly upon the preceding equations
  - the system of equations is manifestly hyperbolic (if the metric is non-singular and maintains a definite signature)
    - the hope is that it would be simple to discretize using standard numerical techniques
  - the ”constraint” equations are the generalized harmonic coordinate conditions
    - simpler to control “constraint violating modes” when present
  - one can view the source functions as being analogous to the lapse and shift in an ADM style decomposition, encoding the 4 coordinate degrees of freedom

Coordinate Issues

- The source functions encode the coordinate degrees of freedom of the spacetime
  - how does one specify H^u to achieve a particular slicing/spatial gauge?
  - what class of evolutions equations for H^u can be used that will not adversely affect the well posedness of the system of equations?

Specifying the spacetime coordinates

- A way to gain insight into how a given H^u could affect the coordinates is to appeal to the ADM metric decomposition

$$\boxed{ds^2 = -\alpha^2 dt^2 + h_{ij}\left(dx^i + \beta^i dt\right)\left(dx^j + \beta^j dt\right)}$$

then

$$\boxed{\begin{aligned} H \cdot n &\equiv H_\mu n^\mu = -n^\mu \partial_\mu \ln \alpha - K \\ \perp H^i &\equiv H_\mu h^{i\mu} = \frac{1}{\alpha} n^\mu \partial_\mu \beta^i + h^{ij} \partial_j \ln \alpha - \overline{\Gamma}^i{}_{jk} h^{jk} \end{aligned}}$$

or

$$\boxed{\begin{aligned} \partial_t \alpha &= -\alpha^2 H \cdot n + \ldots \\ \partial_t \beta^i &= \alpha^2 \perp H^i + \ldots \end{aligned}}$$

Specifying the spacetime coordinates

- Therefore, H^t (H^i) can be chosen to *drive* α (β^i) to desired values
  - for example, the following slicing conditions are all designed to keep the lapse from "collapsing", and have so far proven useful in removing some of the coordinate problems with harmonic time slicing

$$H_t = \xi \frac{\alpha - 1}{\alpha^n}$$

$$\partial_t H_t = \xi \partial_t \left(\frac{\alpha - 1}{\alpha^n} \right)$$

$$\nabla^\mu \nabla_\mu H_t = -\xi \frac{\alpha - 1}{\alpha^n} - \zeta \partial_t H_t$$

Constraint Damping

- Following a suggestion by C. Gundlach (*[C. Gundlach, J. M. Martin-Garcia, G. Calabrese, I. Hinder, gr-qc/0504114]* based on earlier work by Brodbeck et al *[J. Math. Phys. 40, 909 (1999)]*) modify the Einstein equations in harmonic form as follows:

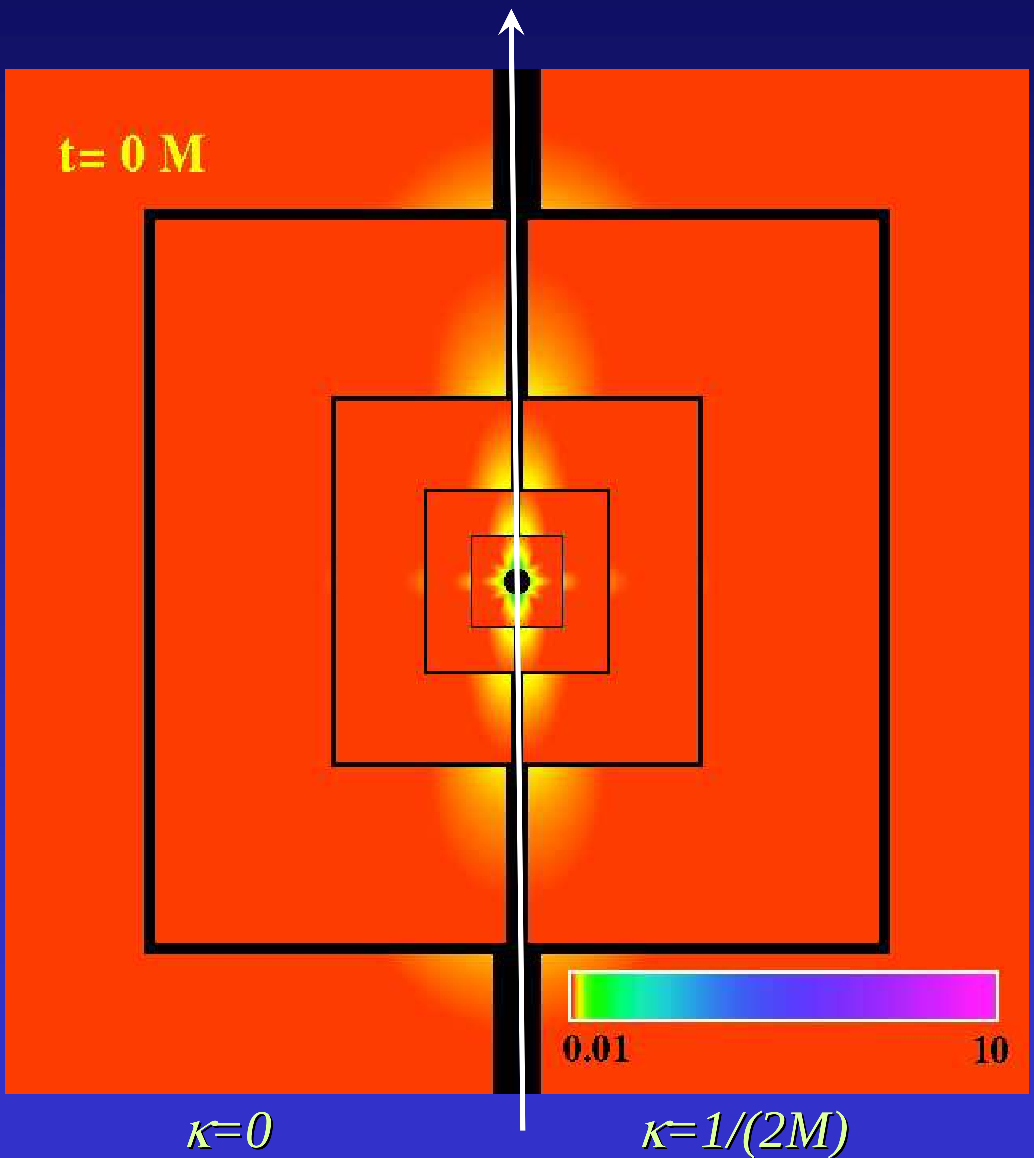
$$\boxed{g^{\alpha\beta} g_{\mu\nu,\alpha\beta} + \ldots + \kappa\left(n_\mu C_\nu + n_\nu C_\mu - g_{\mu\nu} n^\alpha C_\alpha\right) = 0}$$

where

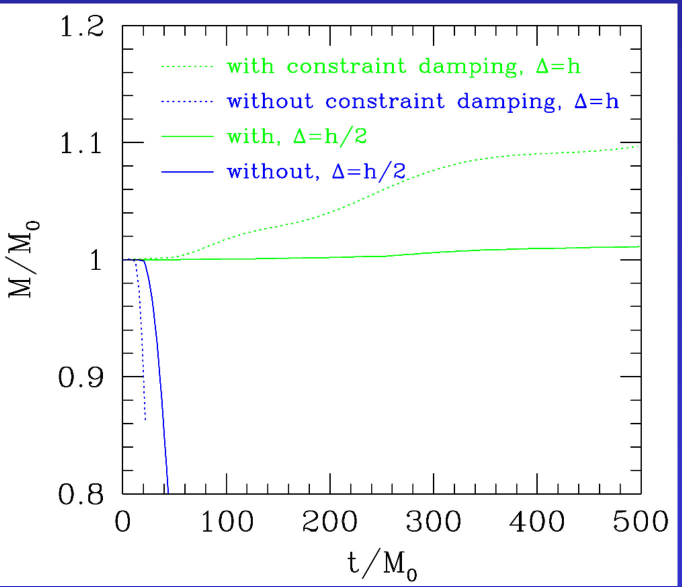
$$\begin{aligned} C^\mu &\equiv H^\mu - \nabla^\alpha \nabla_\alpha x^\mu \\ n_\mu &\equiv -\alpha \nabla_\mu t \end{aligned}$$

- For positive κ, Gundlach et al have shown that all constraint-violations with finite wavelength are damped for linear perturbations around flat spacetime

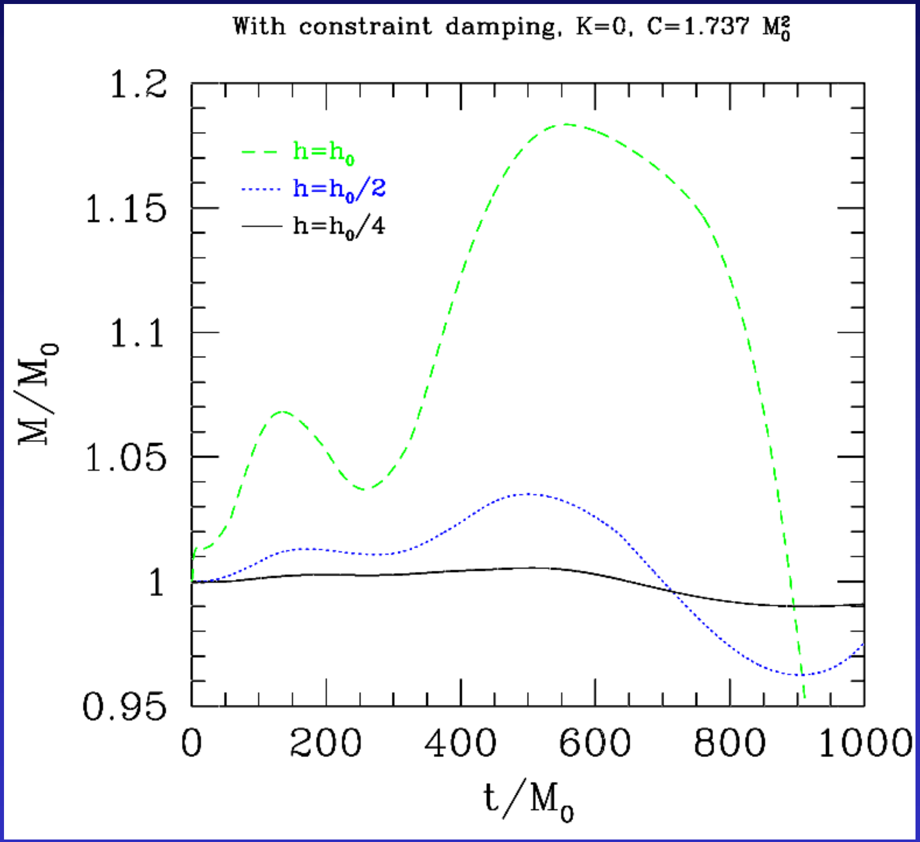
Effect of constraint damping

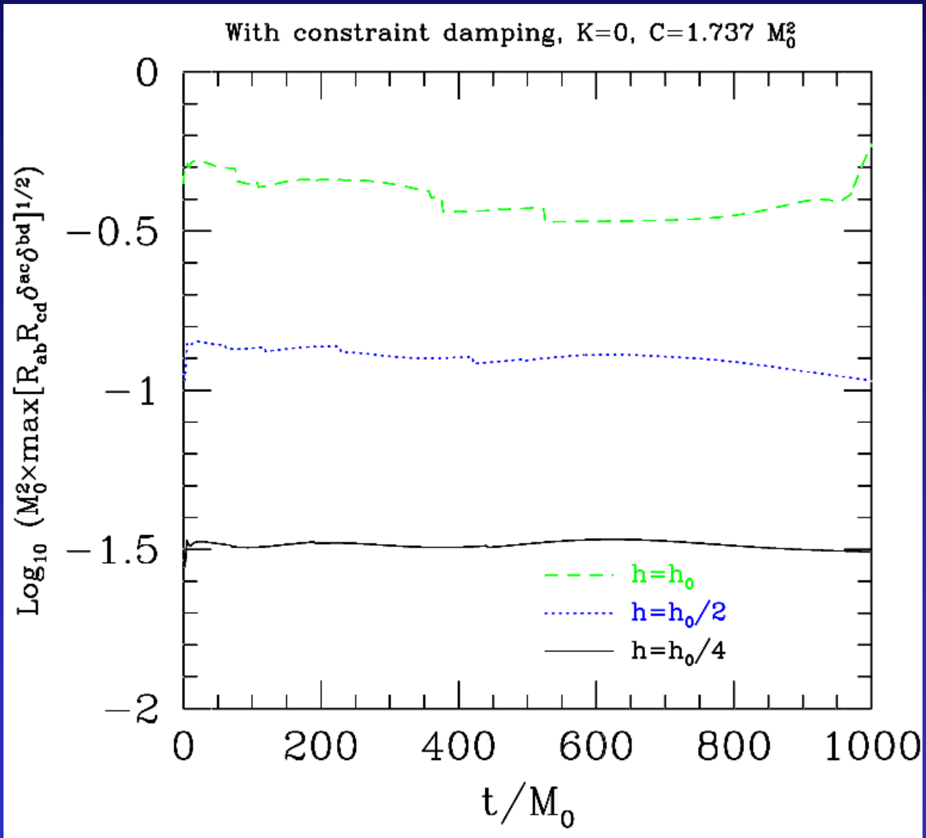


- Axisymmetric simulation of a Schwarzschild black hole, Painleve-Gullstrand coords.
- Left and right simulations use *identical* parameters except for the use of constraint damping



Effect of constraint damping





$$ds^2 = \frac{1}{\alpha^2}dr^2 + r^2 d\Omega^2,\ \ \alpha = \sqrt{1 - \frac{2m}{r} + \frac{C^2}{r^4}},\ \ \beta^r = \frac{C\alpha}{r^2}$$

Merger of a close binary system

- Initial data
  - at this stage I am most interested in the dynamics of binary systems in general relativity, and not with trying to produce an initial set-up that mimics a particular astrophysical scenario
  - hence, use *boosted scalar field collapse* to set up the binary
  - choice for initial geometry and scalar field profile:
    - spatial metric and its first time derivative is conformally flat
    - maximal (gives initial value of lapse and time derivative of conformal factor) and harmonic (gives initial time derivatives of lapse and shift)
    - Hamiltonian and Momentum constraints solved for initial values of the conformal factor and shift, respectively
  - advantages of this approach
    - "simple" in that initial time slice is singularity free
    - all non-trivial initial geometry is driven by the scalar field—when the scalar field amplitude is zero we recover Minkowski spacetime
  - disadvantages
    - ad-hoc in choice of parameters to produce a desired binary system
    - uncontrollable amount of "junk" initial radiation (scalar and gravitational) in the spacetime; though *all* present initial data schemes suffer from this

Merger of a close binary system

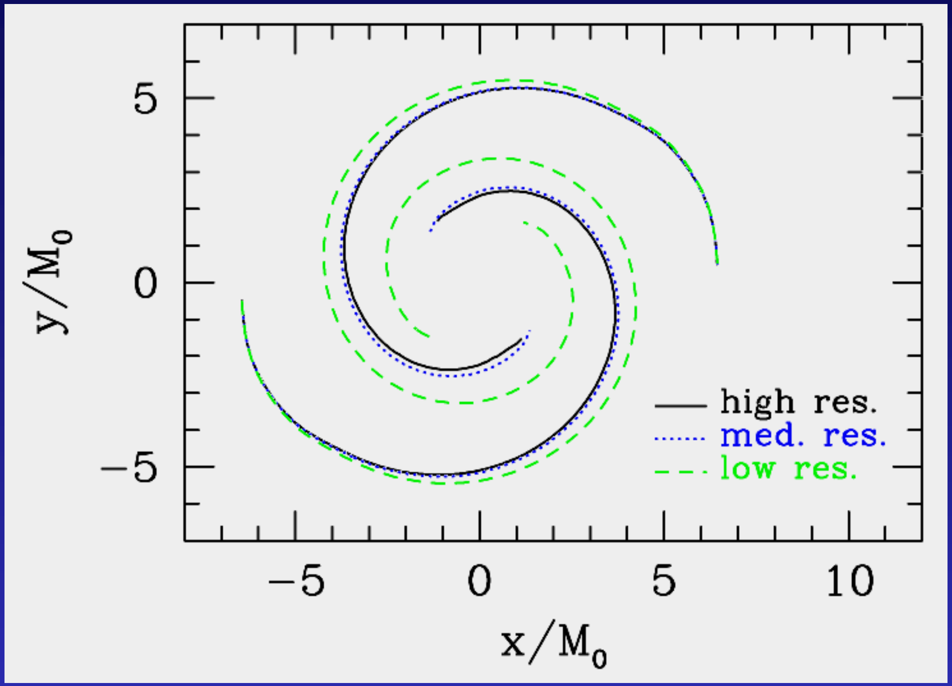
- Gauge conditions:

$$\boxed{\begin{gathered}\nabla^{\mu}\nabla_{\mu}H_t = -\xi\frac{\alpha-1}{\alpha^n} - \zeta\partial_t H_t,\\ H_i = 0\\ \xi \sim 6/M, \zeta \sim 1/M, n = 5\end{gathered}}$$

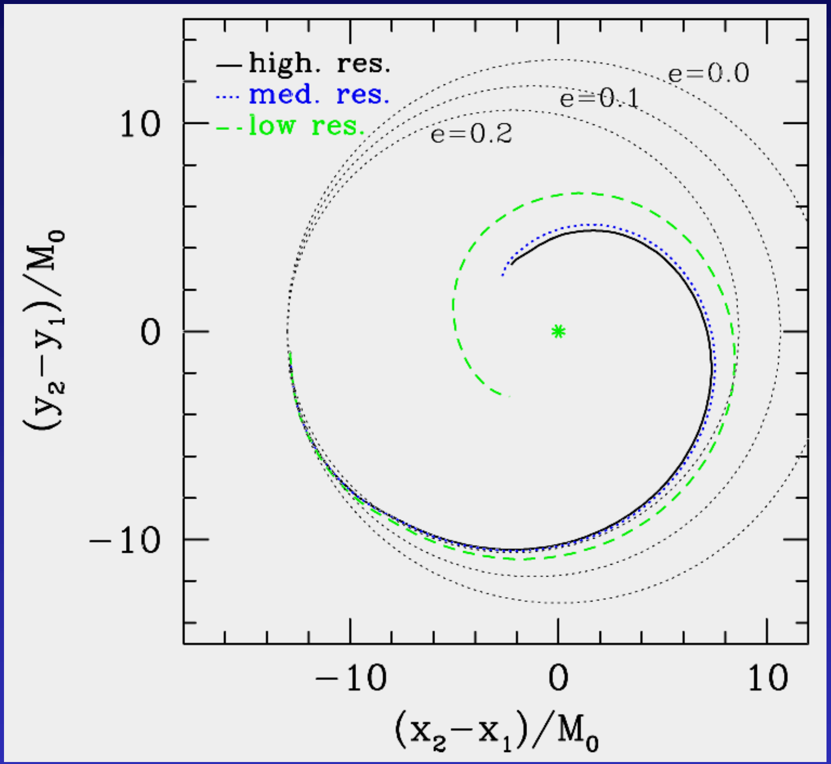
  - Note: this is strictly speaking *not* spatial harmonic gauge, which is defined in terms of the "vector" components of the source function

- Constraint damping term $\boxed{\kappa \sim 1/M}$

Orbit



Simulation (center of mass) coordinates

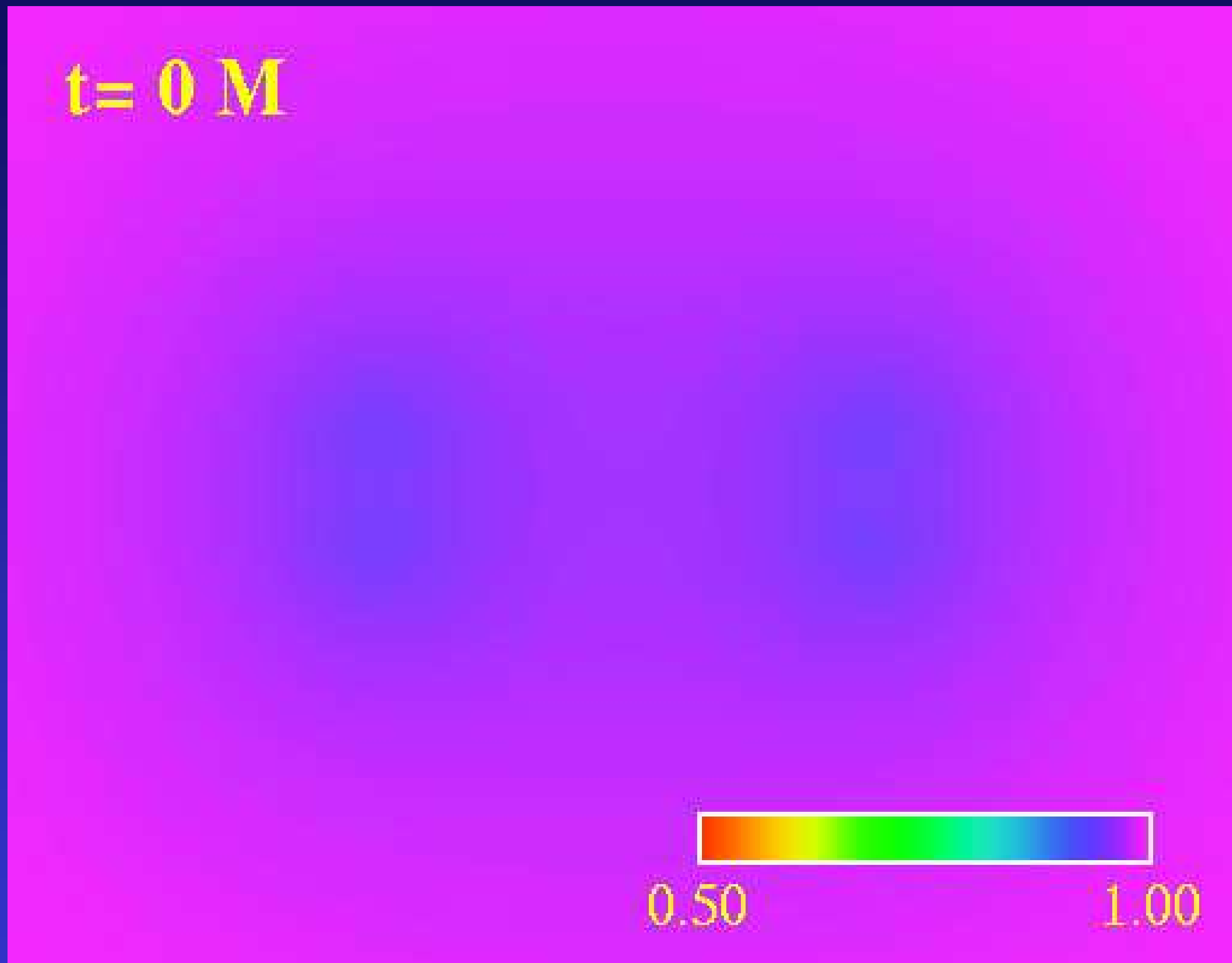


Reduced mass frame; heavier lines are position of BH 1 relative to BH 2 (green star); thinner black lines are reference ellipses

- Initially:
  - equal mass components
  - eccentricity $e \sim 0 - 0.2$
  - coordinate separation of black holes ~ 13M
  - proper distance between horizons ~ 16M
  - velocity of each black hole ~0.16
  - spin angular momentum = 0
  - ADM Mass ~ 2.4M

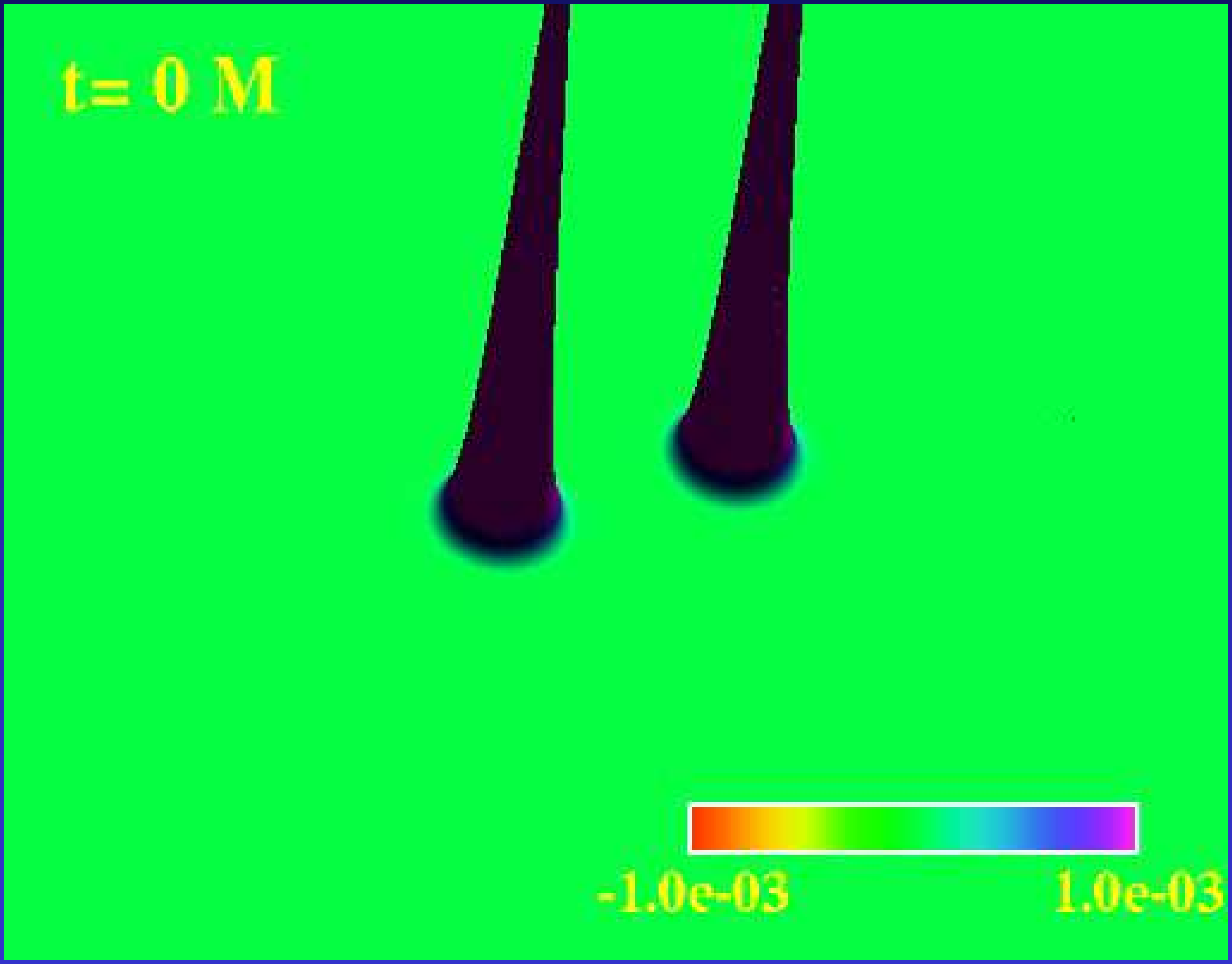
- Final black hole:
  - $M_f \sim 1.9M$
  - Kerr parameter a ~ 0.70
  - error ~ 5%

Lapse function α, orbital plane



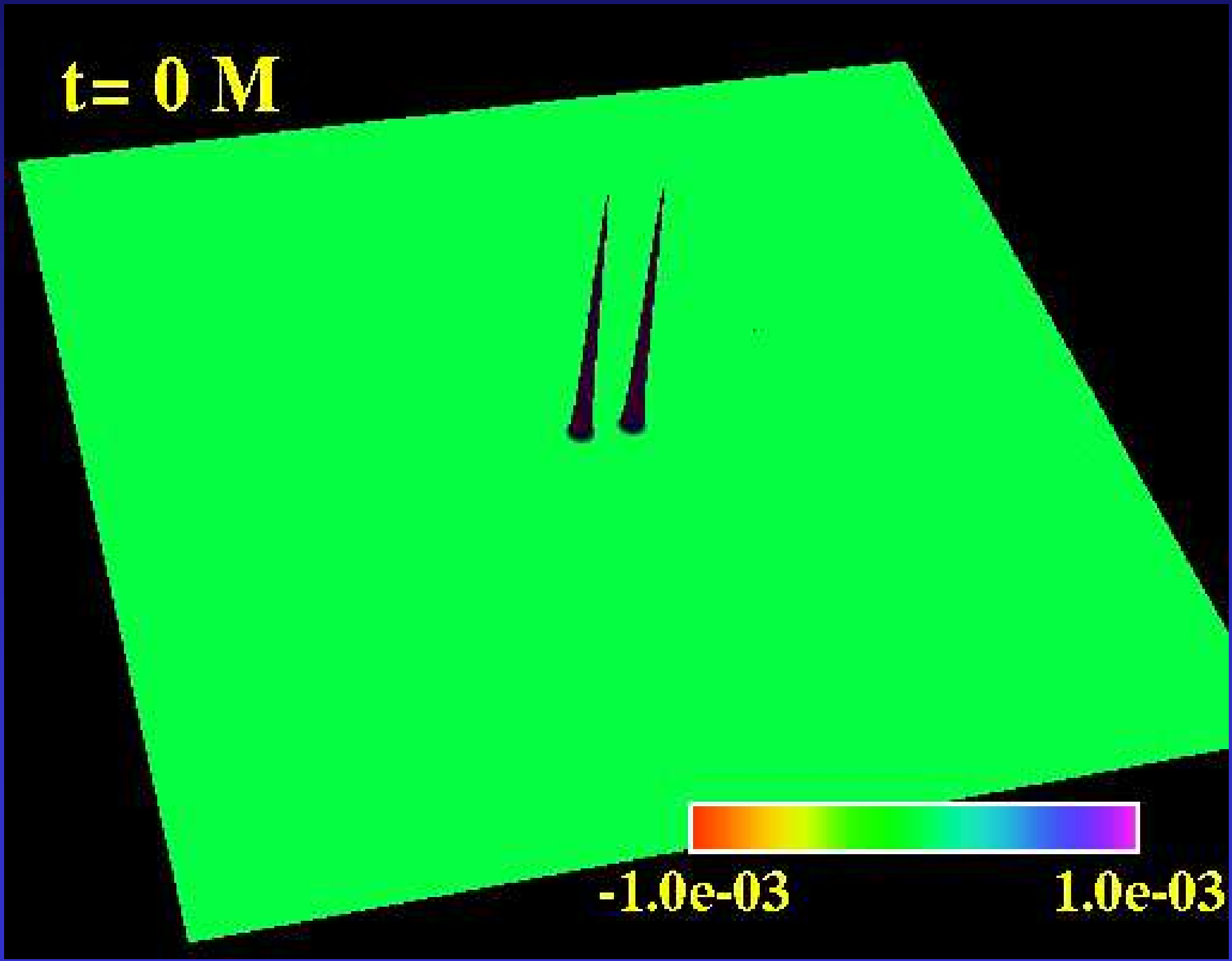
All animations: time in units of the mass of a single, initial black hole, and from medium resolution simulation

Scalar field $\phi.r$, uncompactified coordinates



Scalar field $\phi.r$, compactified (code) coordinates

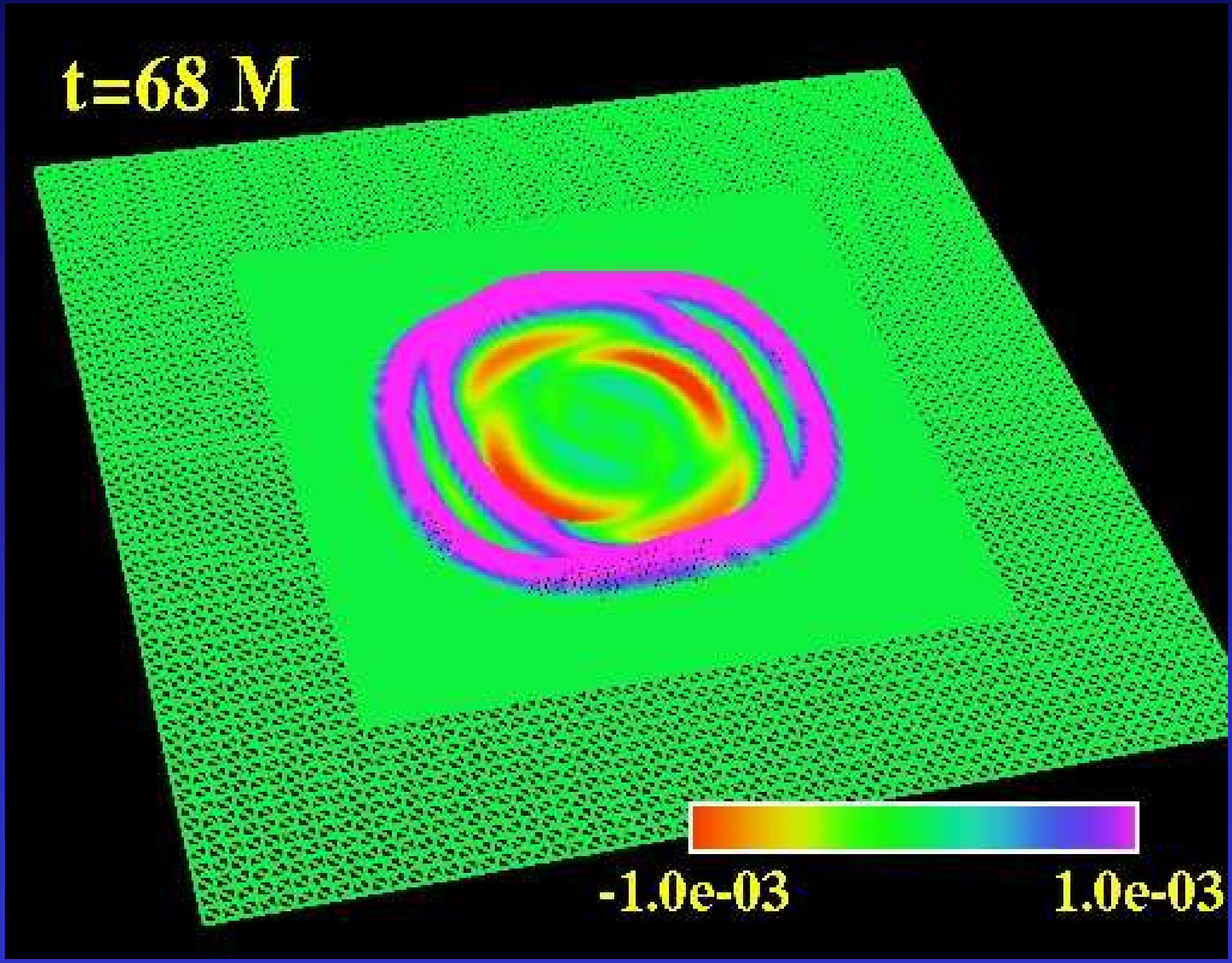
$$\bar{x} = \tan(x\pi/2), \bar{y} = \tan(y\pi/2), \bar{z} = \tan(z\pi/2)$$



Summary of computation – medium resolution simulation

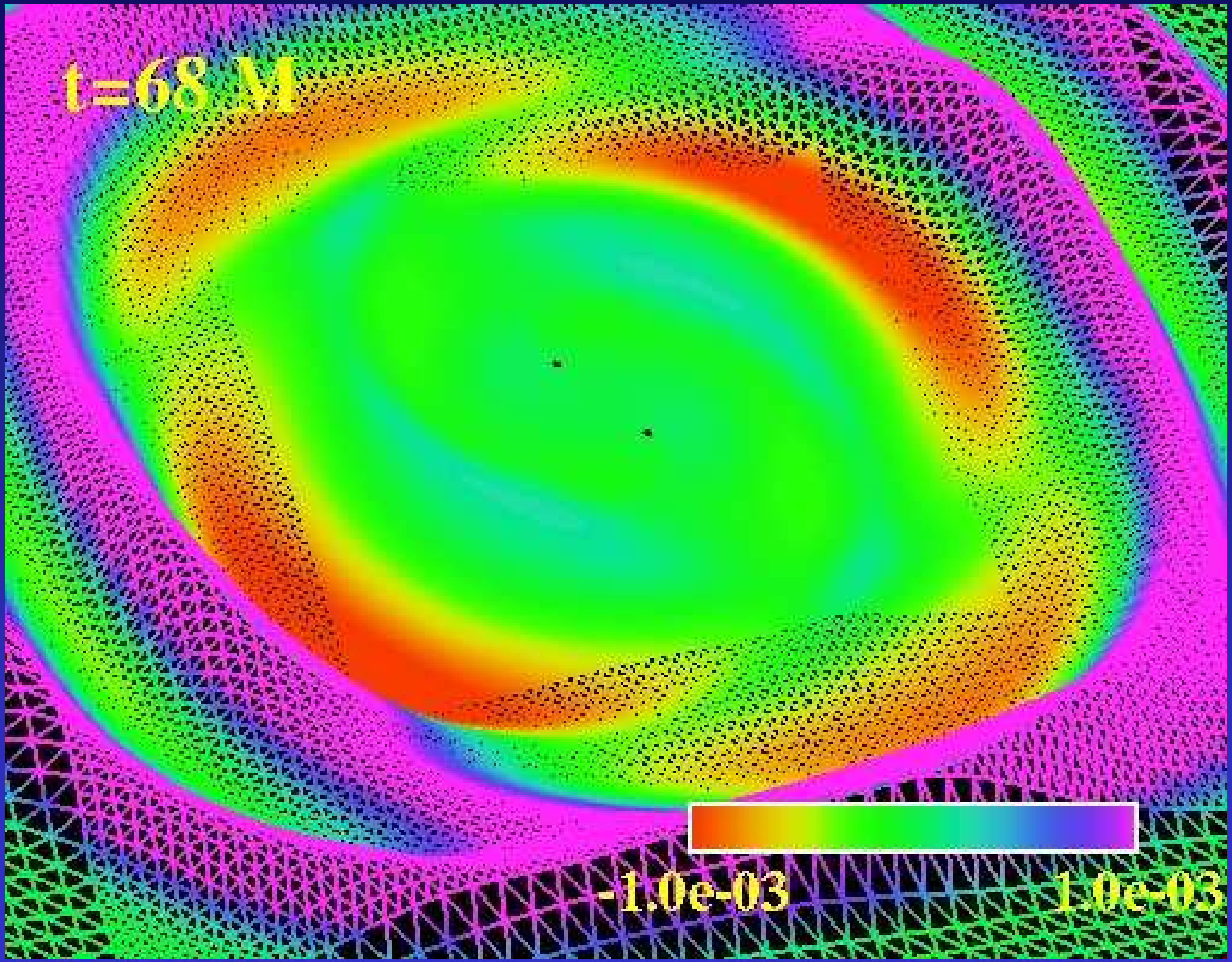
- base grid resolution 33^3
  - 9 levels of 2:1 mesh refinement (*effective* finest grid resolution of 8192^3) … switched to 8 levels maximum at around 150M
  - ~ 50,000 time steps on finest level
  - ~550 hours on 48 nodes of UBC's vnp4 Xeon cluster (26,000 CPU hours total)
  - maximum total memory usage ~ 10GB, disk usage ~ 100GB (and this is very infrequent output!)

Sample mesh structure



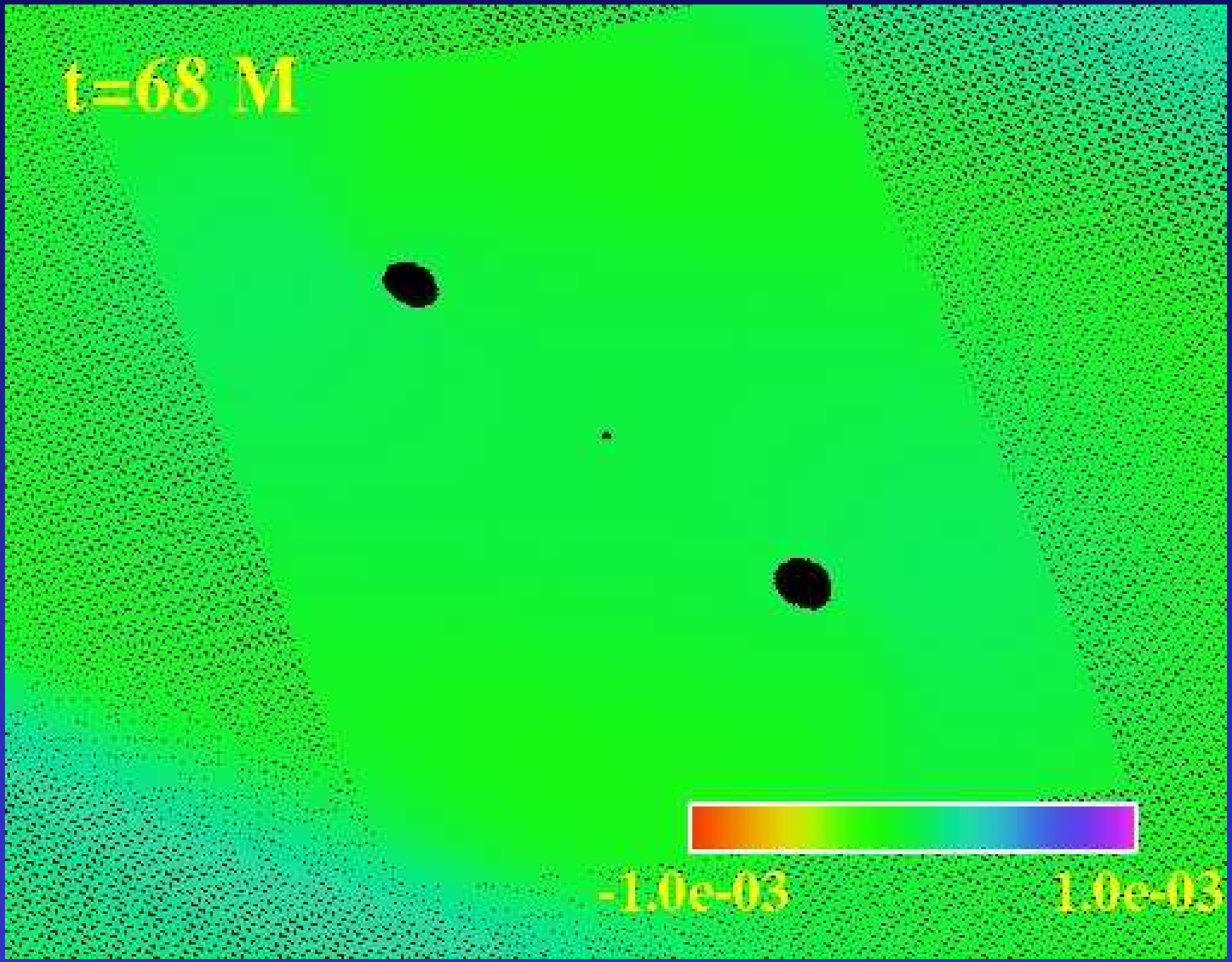
Scalar field ϕ . r, z=0 slice

Sample mesh structure



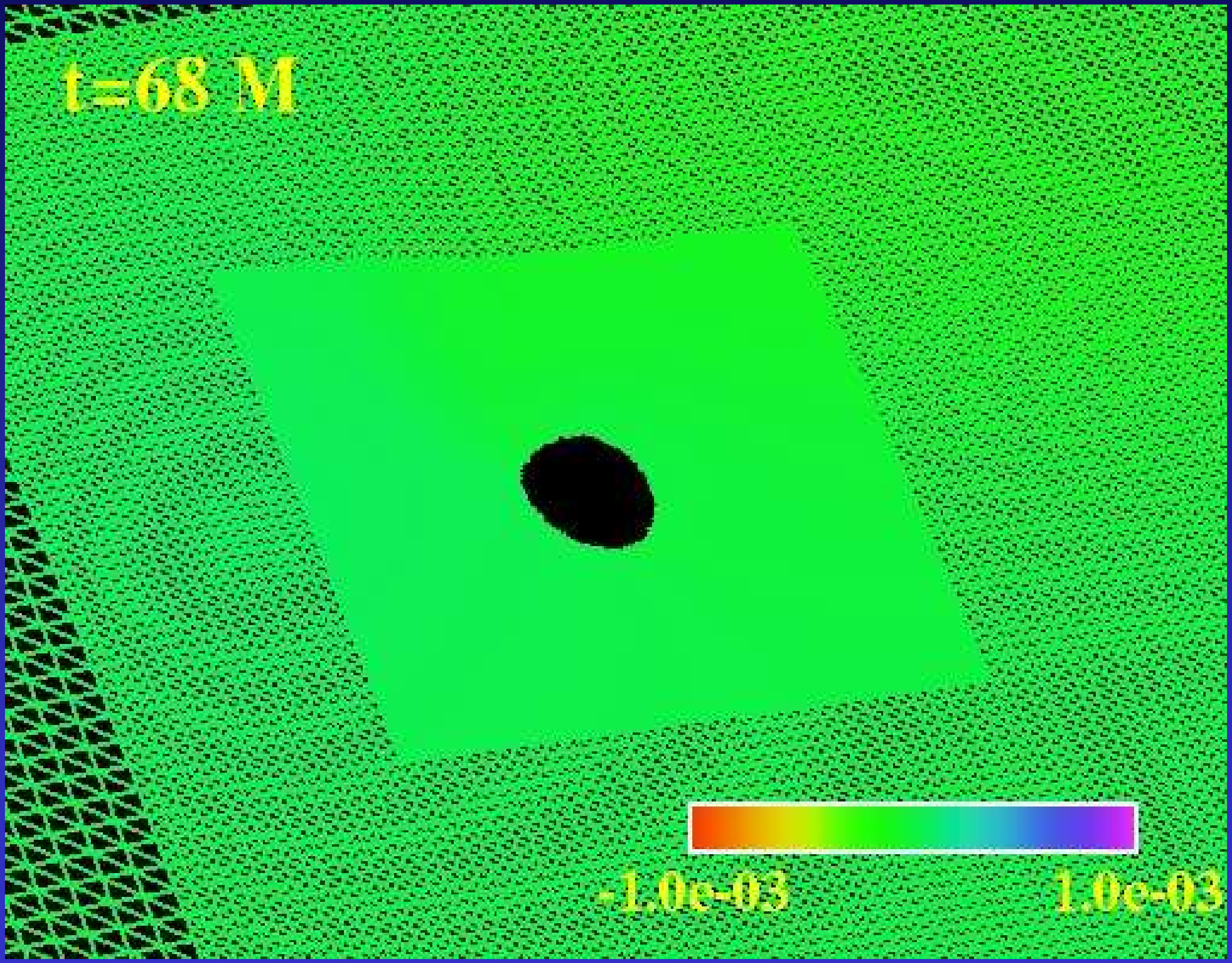
Scalar field ϕ . $r, z=0$ slice

Sample mesh structure



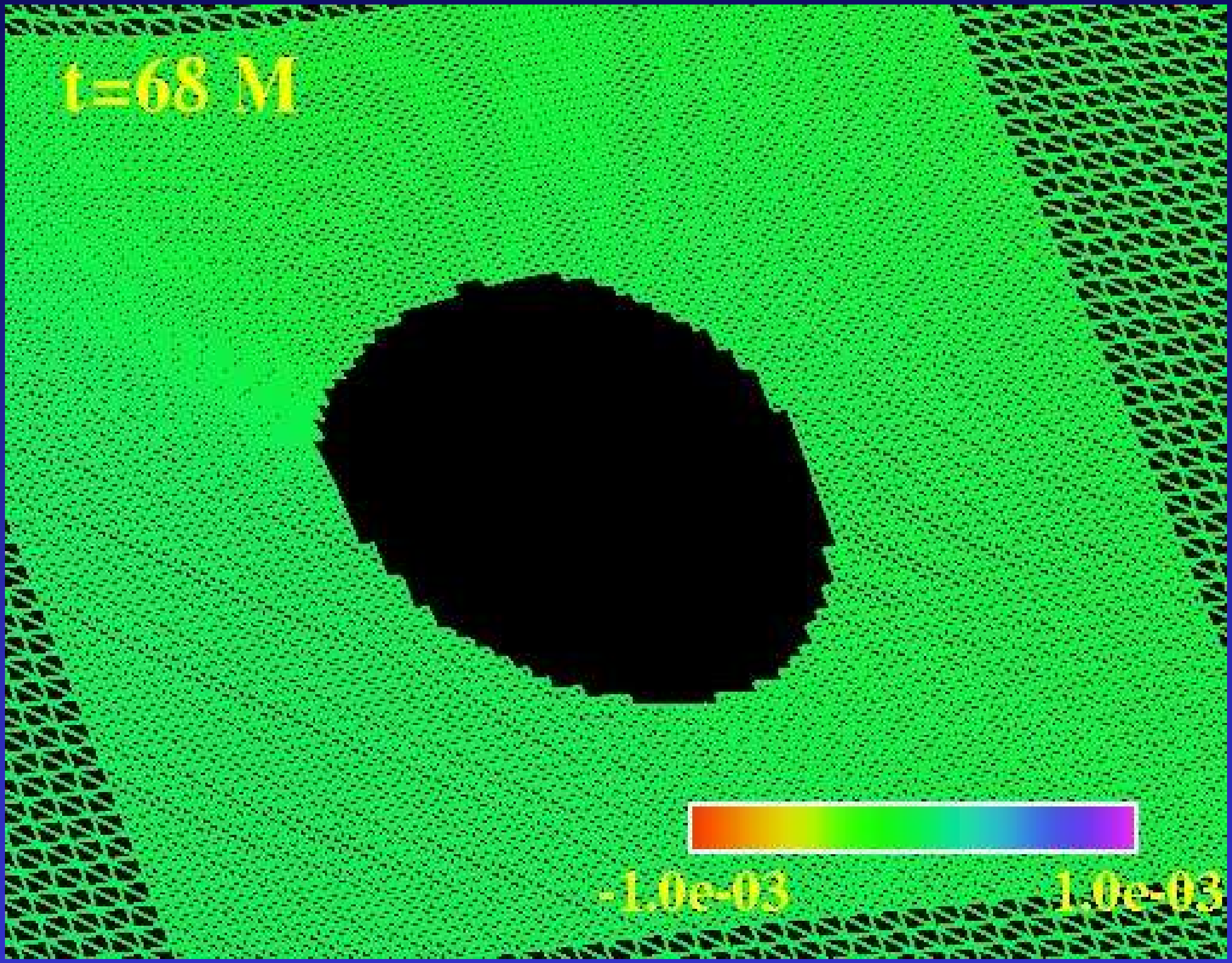
Scalar field ϕ . r, $z=0$ slice

Sample mesh structure



Scalar field $\phi \cdot r$, $z=0$ slice

Sample mesh structure

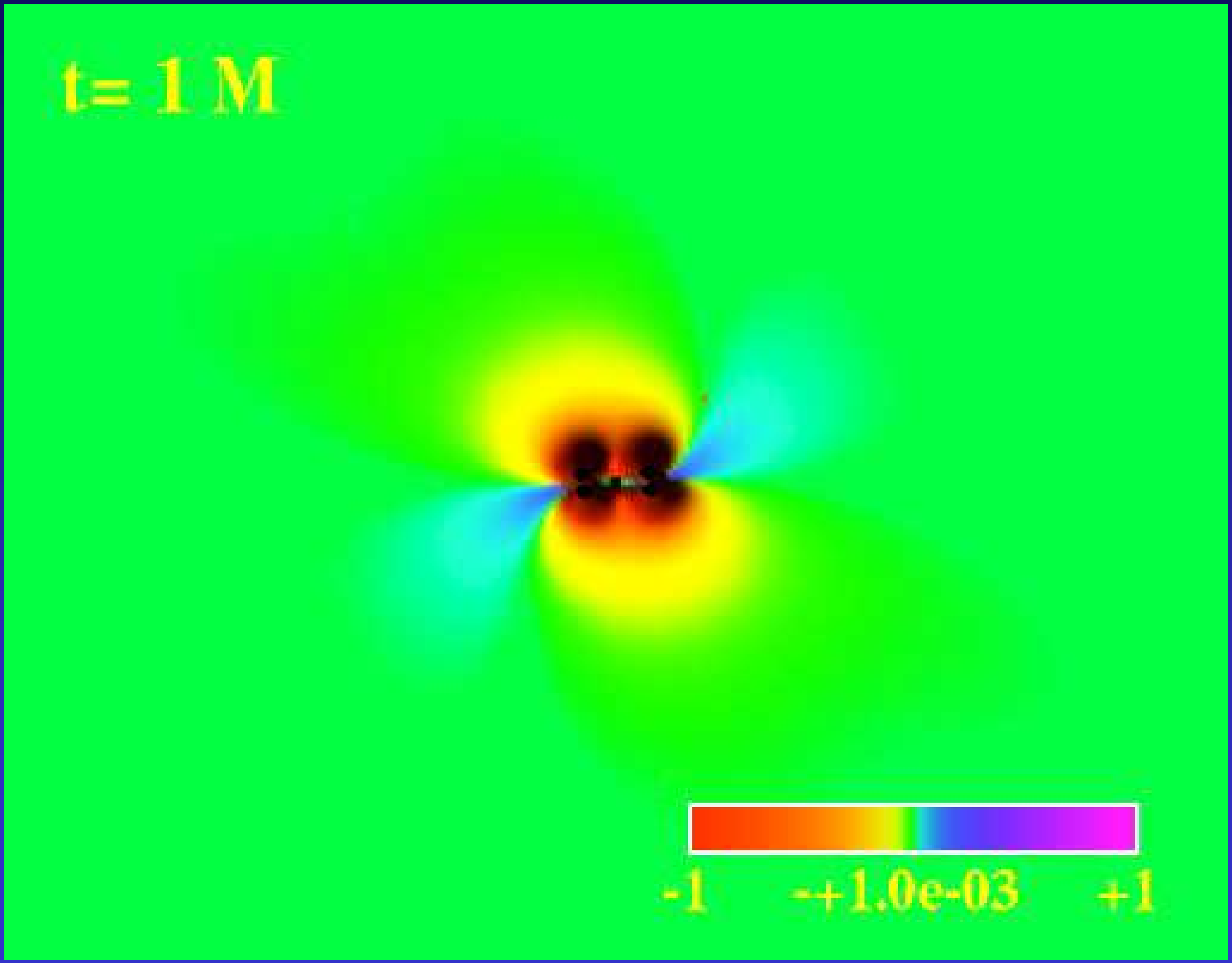


Scalar field ϕ . $r, z=0$ slice

Waveform extraction

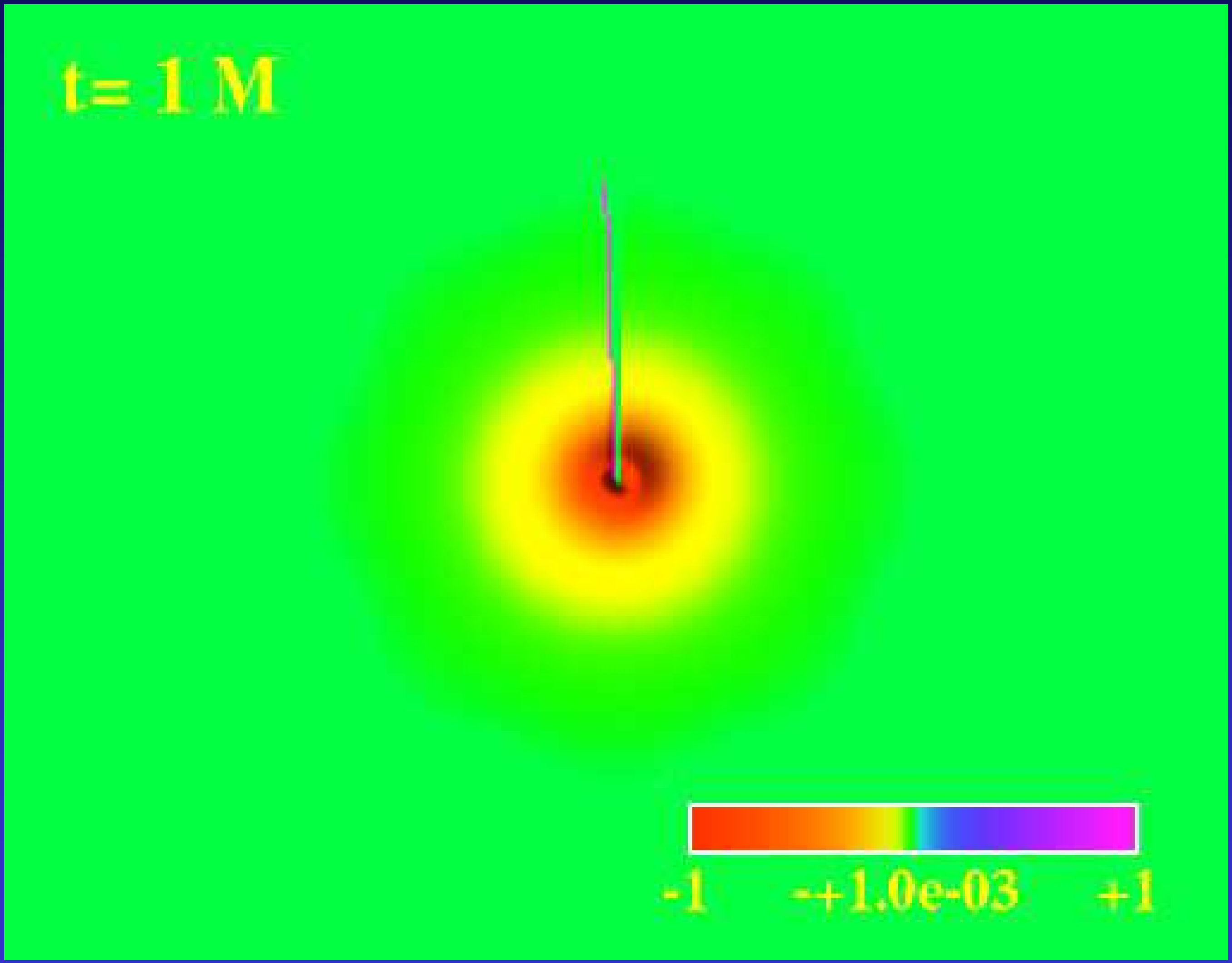
- Can we extract a waveform in light of
  - unphysical radiation in initial data
  - Compactification; i.e. poor resolution near outer boundaries
  - AMR "noise": finding the waveform typically requires taking derivatives of metric functions; enhances noise
- Answer seems to be *yes*, though the caveat is how accurately does one need the waveform.

Waveform extraction



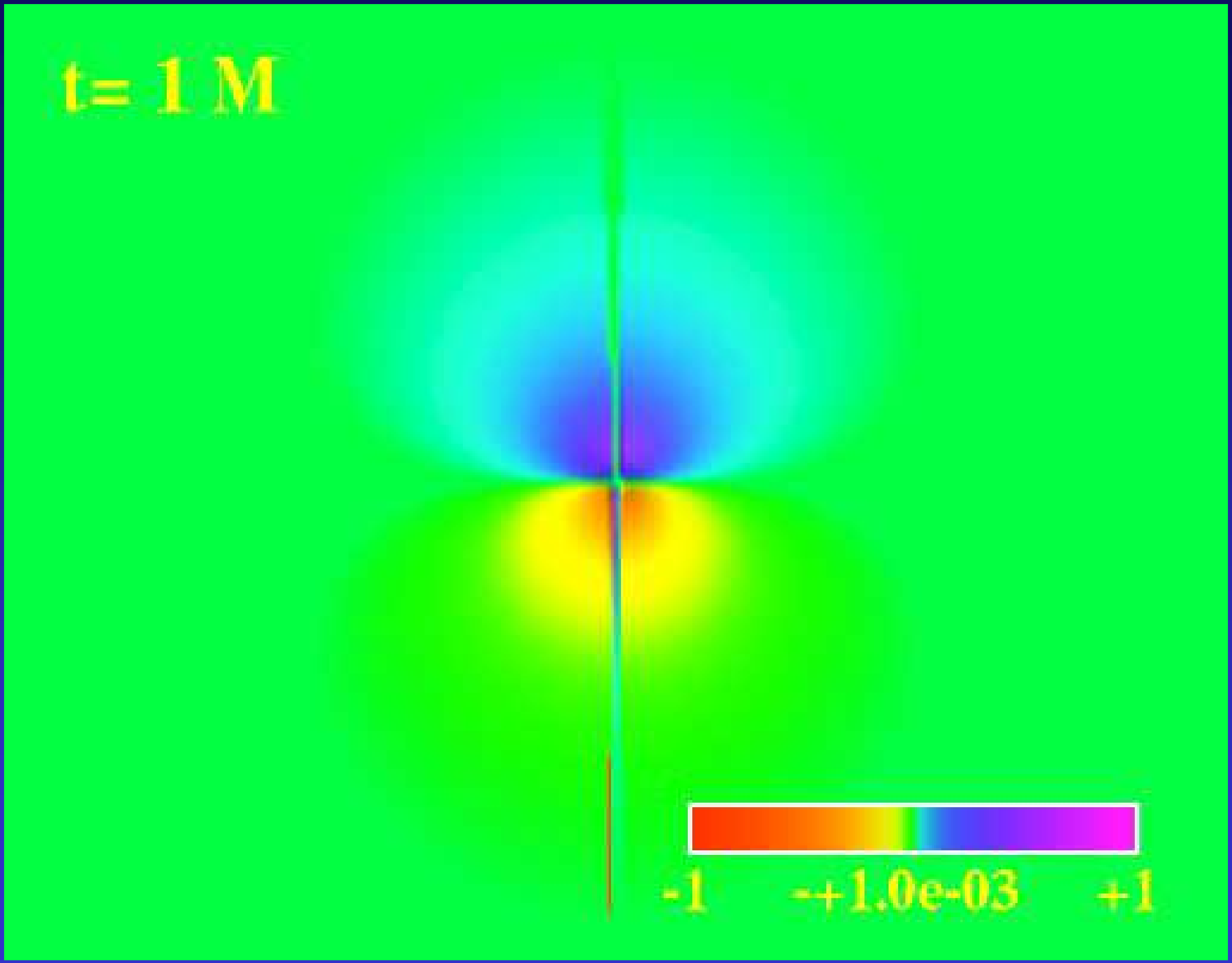
Real component of the Newman-Penrose scalar Ψ_4 times r, $z=0$ slice of the solution

Waveform extraction



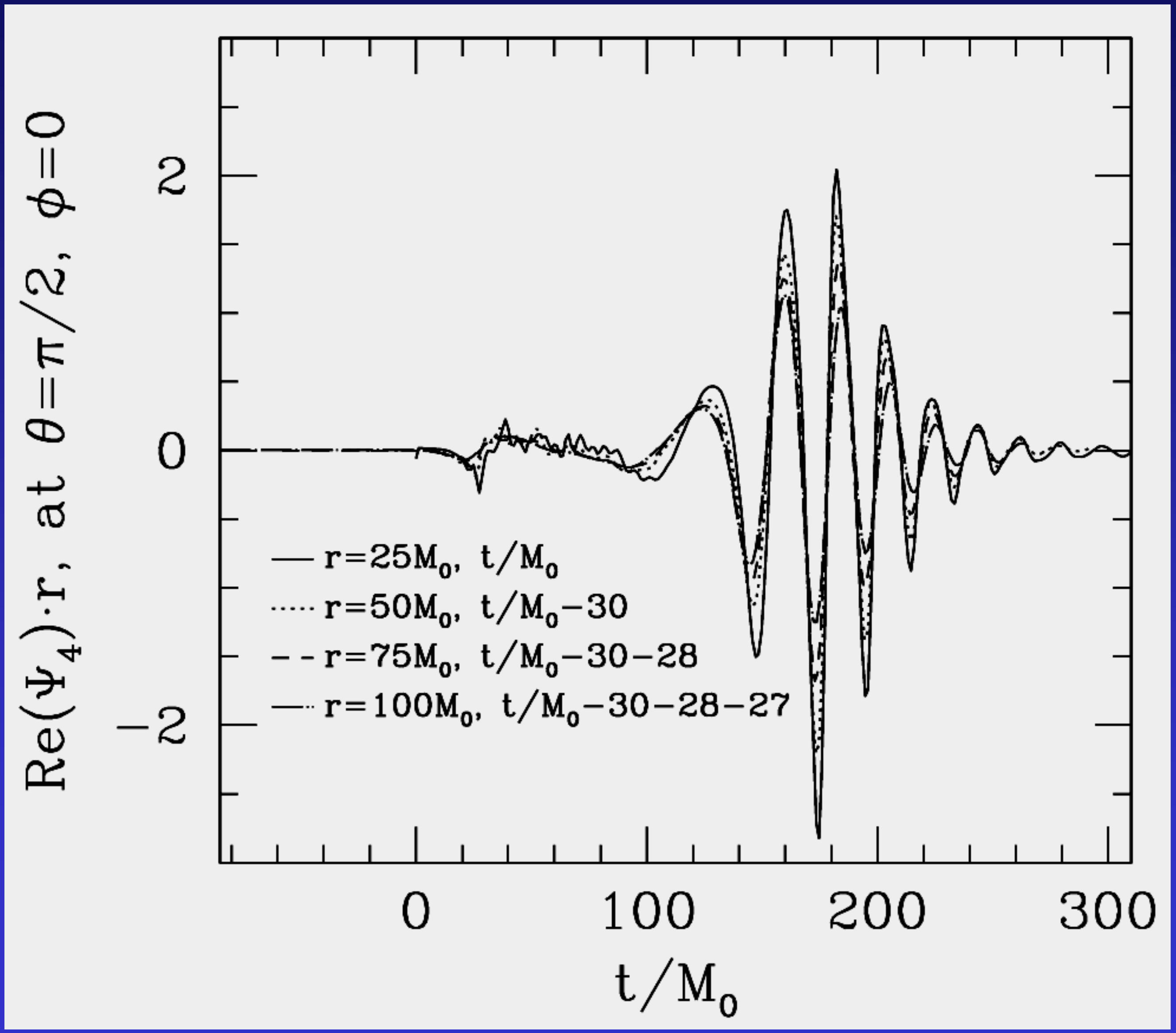
Real component of the Newman-Penrose scalar Ψ_4 times r, $x=0$ slice of the solution

Waveform extraction



Imaginary component of the Newman-Penrose scalar Ψ_4 times r, $x=0$ slice of the solution

Waveform extraction



Energy radiated ?

- On some sphere of radius R, a large distance from the source:

$$\frac{dE}{dt} = \frac{R^2}{4\pi}\int p(t,\theta,\phi)\,d\Omega$$

$$p(t,\theta,\phi) = \int_0^t \Psi_4(t',\theta,\phi)\,dt' \cdot \int_0^t \Psi_4^*(t',\theta,\phi)\,dt'$$

Difficult to integrate *accurately* from a numerical simulation:

- *R*=25*M* : 4.7% (% relative to 2M)
  R=50*M* : 3.2%
  R=75*M* : 2.7%
  R=100*M* : 2.3%

- Other estimates:
  - Horizon mass : 5%
  - From comparison of wave amplitudes from boosted, head-on collision with similar simulation parameters, and known estimates from the literature, also suggests total is around 5% *[Hobill et al, PRD 52, 2044 (1995)]*;

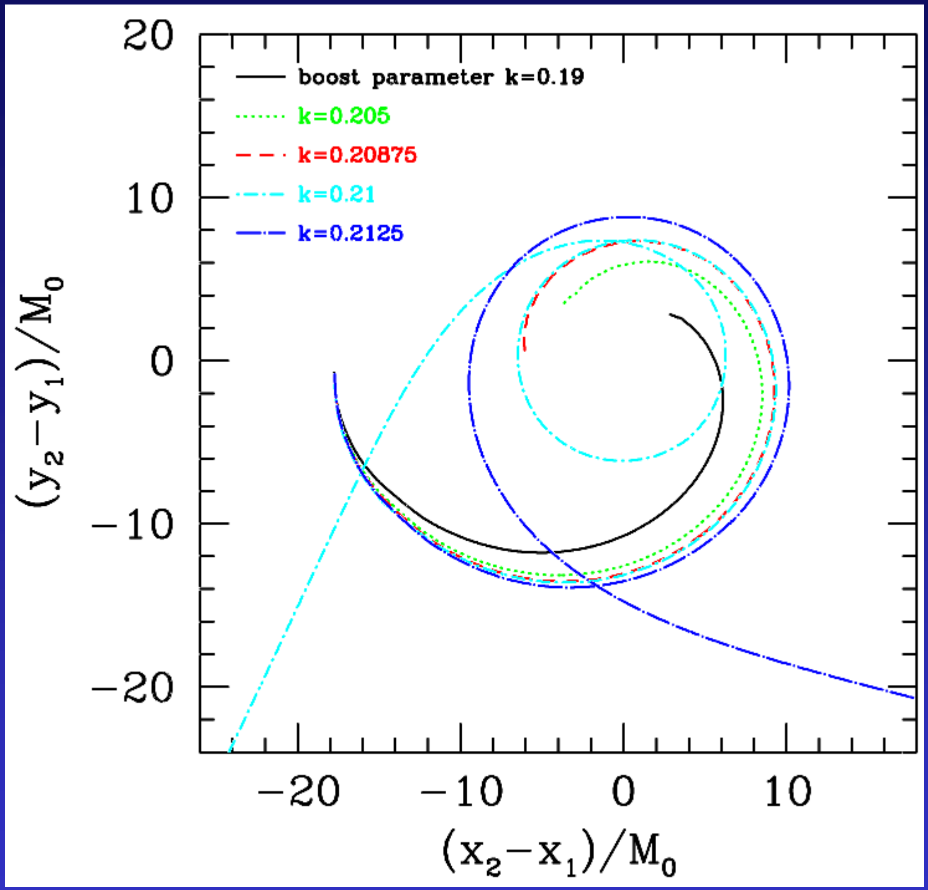
What does this wave represent?

- Scale the system to M_0 = 10 solar masses ~ 2×10^{31} kg
  - radius of each black hole in the binary is ~ 30km
  - radius of final black hole is ~ 55km
  - distance where the wave was extracted from the final black hole ~ 750km
  - frequency of the wave ~ 600Hz
  - fractional oscillatory "distortion" in space induced by the wave transverse to the direction of propagation has a maximum amplitude $\Delta L/L \sim 3\times10^{-3}$
    - a 2m tall person will get stretched/squeezed by ~ 6mm as the wave passes
    - LIGO's arms would change ~ 12m. Wave amplitude decays like 1/distance from source; e.g. at 10Mpc the change in arms ~ 3×10^{-18}m (which is in the ballpark of what LIGO is trying to measure!!)
  - despite the seemingly small amplitude for the wave, the energy it carries is enormous — around 3×10^{30} kg c^2 ~ 2×10^{46} J ~ 2×10^{54} ergs

Not-so-close binaries

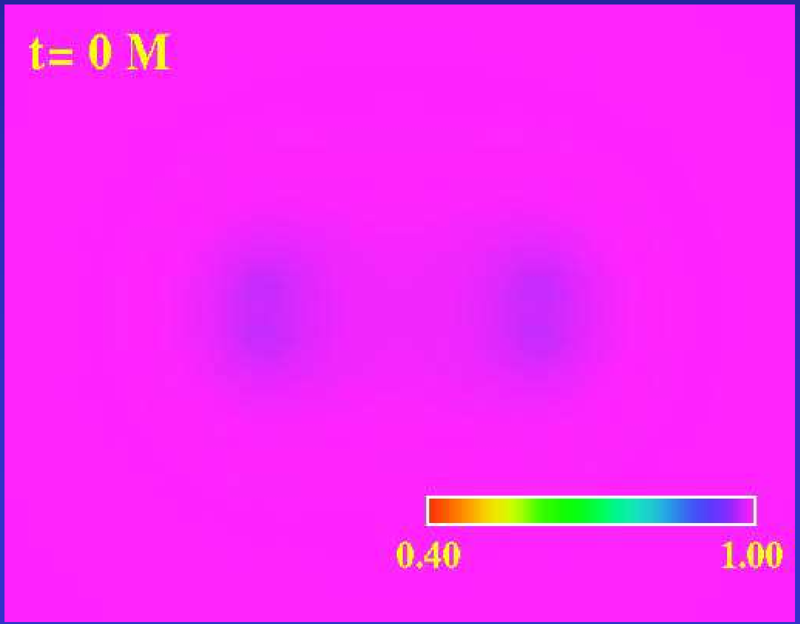
- A couple of questions
  - the waveform seems to be dominated by the collision/ringdown phase of the orbit. Is this generic? i.e. will the last few cycles of a waveform carry away as much as 5% of the energy of the binary?
  - how generic is this plunge/ringdown signal to changes in initial conditions?

Not-so-close binaries

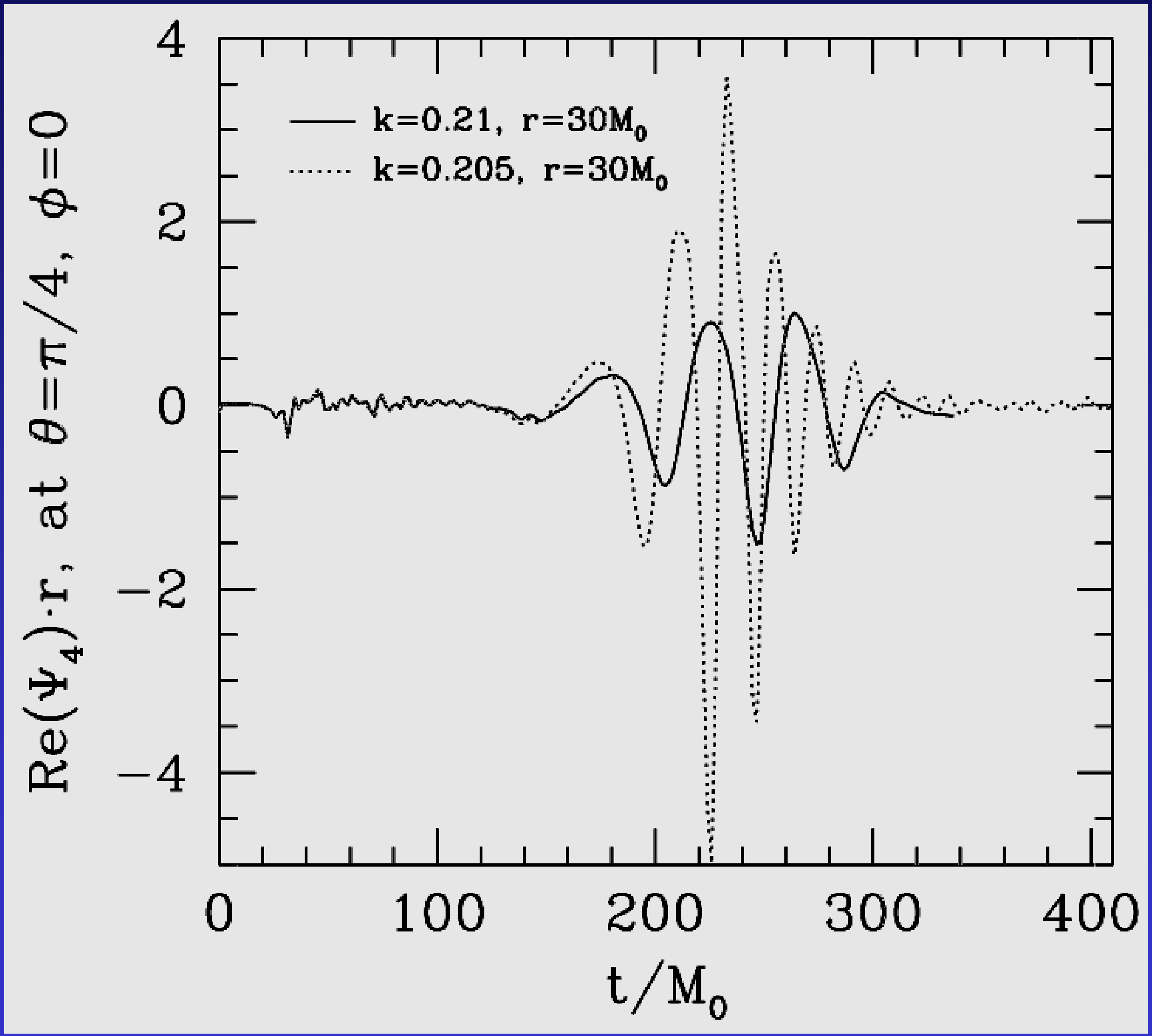


- Initially:
  - equal mass components
  - proper distance between horizons ~ 22M
- NOTE:
  - Simulations do *not* have identical gauge/evolution parameters

Lapse function α, k=.21 example, orbital plane



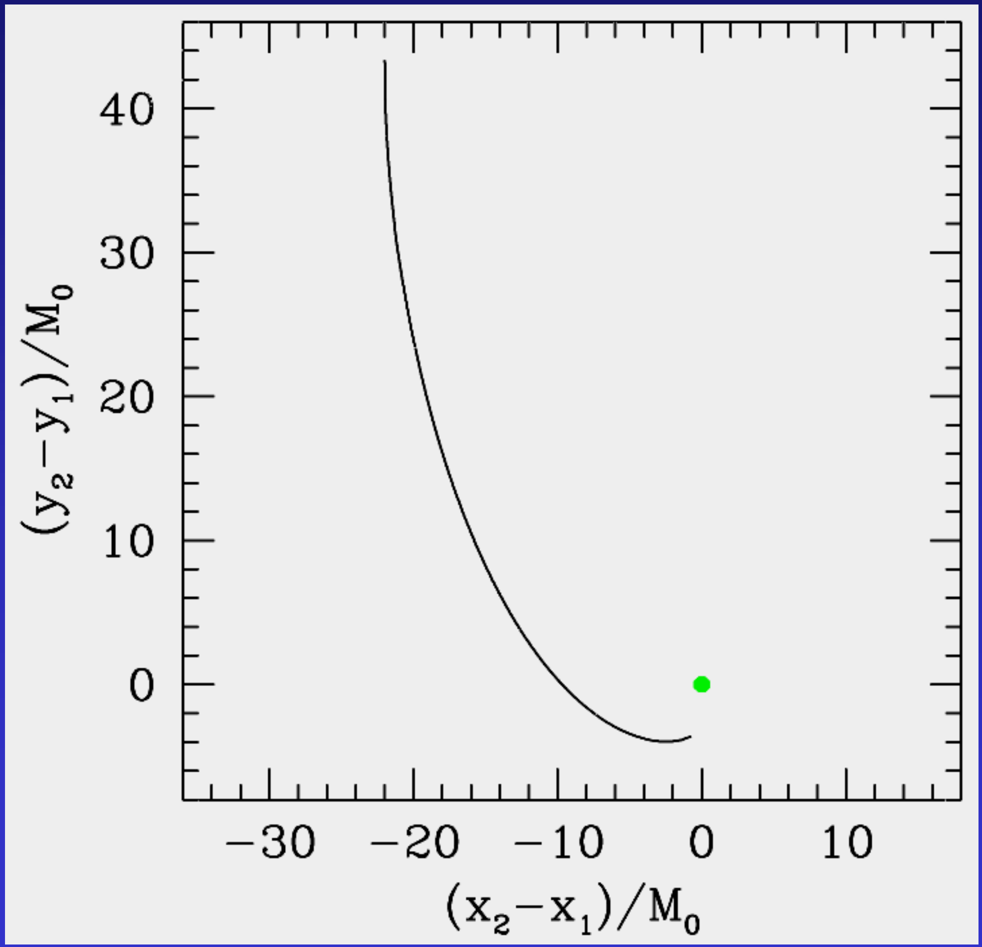
Waveforms

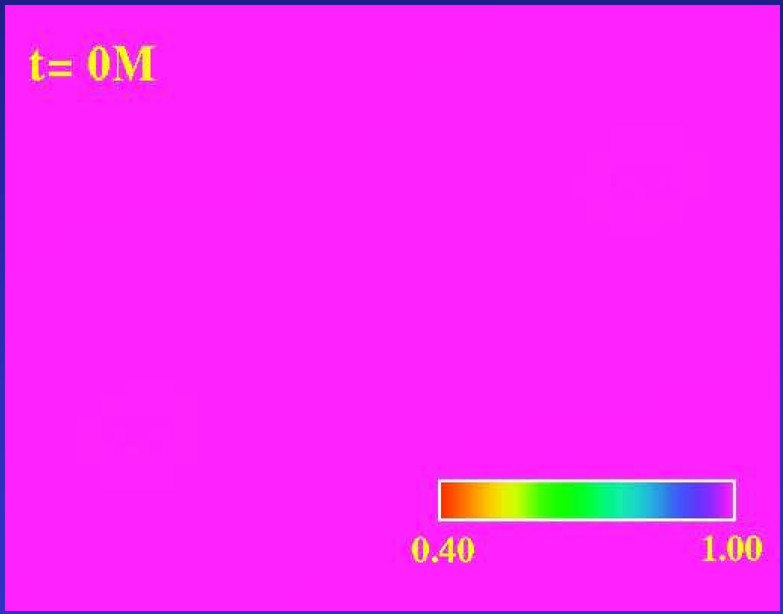


- Total energy radiated, measured at *r=30M*, integrating till *t=320M*
  - *k=.21* : 4%
  - *k=.205* : 5.7%
- So, ~3-4% energy in plunge-ring down??
- NOTE:
  - not convergence tested, though previous single orbit example suggests "low" resolution simulations over-estimate energy by ~20%
  - the fact that the two orbit case starts to separate could be entirely due to numerical error; even so, the hint at ratio of energies of orbit/plunge-ring down may be more robust

Genericity of waveforms?

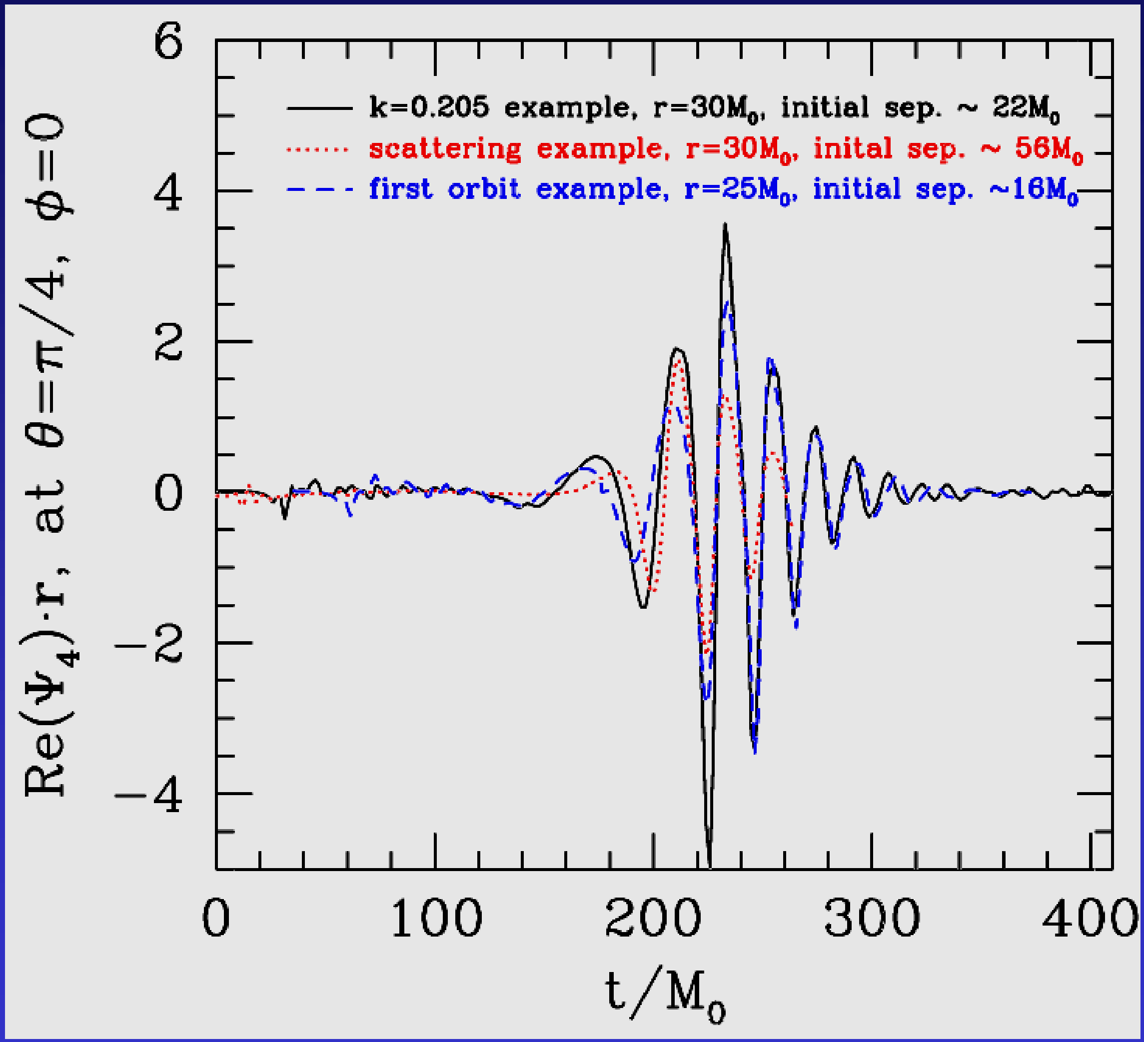
Merger from “scattering” initial data





Lapse function α, orbital plane

Waveforms



Note: different M_0's, and shifted in time

Summary -- near future work

- What physics can one hope to extract from these simulations over the next year or so?
  - very broad initial survey of the qualitative features of the last stages of binary mergers
    - pick a handful of orbital parameters (mass ratio, eccentricity, initial separation, individual black hole spins) widely separated in parameter space
      - computational requirements make it completely impractical to try to come up with a template bank for LIGO at this stage (ever?)
    - try to understand the general features of the emitted waves, the total energy radiated, and range of final spins as a function of the initial parameters, etc.