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Binary Neutron Stars: Helical Symmetry and Waveless Approximation

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I. EINSTEIN EULER SYSTEM

II. HELICAL SYMMETRY AND WAVELESS APPROXIMATION

III. STATIONARY AND QUASI-STATIONARY EQUILIBRIA FOR ROTATING STARS

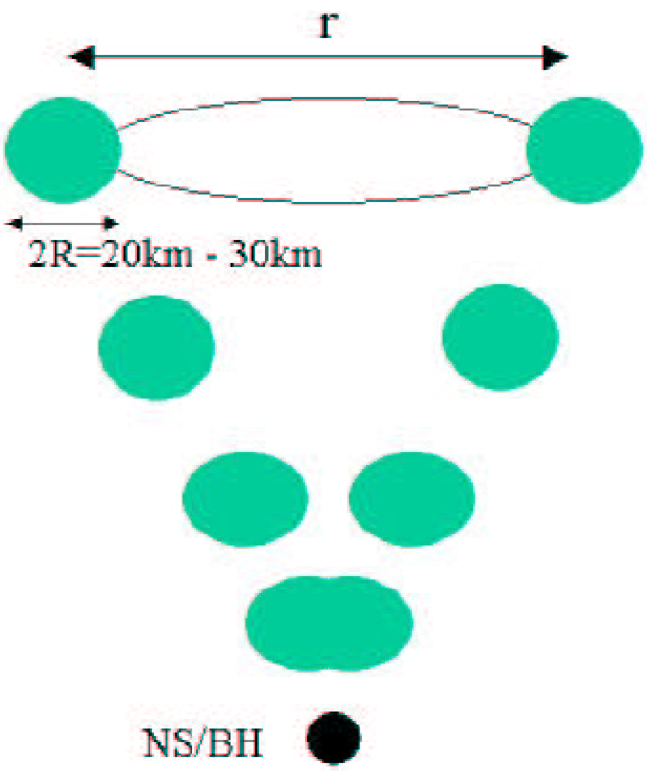
IV. RESULTS FROM EVOLUTION

A. PROMPT COLLAPSE VS HYPERMASSIVE NS

B. GW FROM NS OSCILLATIONS

C. J/M^2 FOR FINAL BH

The last 3 minutes



Inspiral

$f \sim 10 \to\sim 500$ Hz

Post-Newtonian

Point Particle

$r \gg R$, $t_{\mathrm{GW}} \gg P_{\mathrm{orb}}$

Intermediate

$f \gtrsim 500$ Hz

$r \lesssim 4R$, $r \gtrsim 10M$,

$t_{\mathrm{GW}} \gg P_{\mathrm{orb}}$

Finite size NS.

Merger

$f > 1$kHz.

Event rate : 1/yr in $\sim 50 - 100$ Mpc (Kalogera et al. 2004).

$\sim 40 - 600$ events/yr for Advanced LIGO.

I. EINSTEIN-EULER SYSTEM: PERFECT FLUID SPACETIMES

With $q^{\alpha\beta} = g^{\alpha\beta} + u^\alpha u^\beta$ the projection $\perp u^\alpha$,

$$T^{\alpha\beta} = \varepsilon u^\alpha u^\beta + p q^{\alpha\beta} = (\varepsilon + p) u^\alpha u^\beta + p g^{\alpha\beta}.$$

Binary NS inspiral is modeled by a perfect-fluid spacetime, a spacetime M,g whose whose metric satisfies

$$G_{\alpha\beta} = 8\pi T_{\alpha\beta},$$

with $T^{\alpha\beta}$ a perfect-fluid energy-momentum tensor.

$$T^{\alpha\beta} = \varepsilon u^\alpha u^\beta + p q^{\alpha\beta},$$

$$q^{\alpha\beta} = g^{\alpha\beta} + u^\alpha u^\beta$$

Barotropic flows: enthalpy and injection energy

A fluid with a one-parameter EOS is called *barotropic*. Neutron star matter is accurately described by a one-parameter EOS because it is approximately *isentropic:* Neutron stars rapidly cool far below the Fermi energy (10^{13}K » m_p), effectively to zero temperature and entropy.

(There is, however, a composition gradient in neutron stars, with the density of protons and electrons ordinarily increasing outward, and this dominates a departure from a barotropic equation of state in stellar oscillations).

1-PARAMETER EOS

$$\varepsilon = \varepsilon(p) \implies \frac{\nabla p}{\varepsilon + p} = \nabla \ln h,$$

$$h := \exp\left[\int_0^p \frac{dp}{\varepsilon + p}\right].$$

Then the Euler eqn $u^\beta \nabla_\beta u^\alpha = -q^{\alpha\beta} \frac{\nabla_\beta p}{(\varepsilon + p)}$ becomes

$$u^\beta \nabla_\beta u^\alpha = -q^{\alpha\beta} \nabla_\beta \ln h.$$

Introducing h allows one to find a first integral of the equation of hydrostatic equilibrium for corotating and irrotational binaries stationary in a rotating frame: with helical killing vector k^α

For corotation, $u^\alpha = u^t k^\alpha$, hydrostatic equilibrium

takes the form $$0 = -\nabla_\alpha \ln u^t + \nabla_\alpha \ln h = \nabla_\alpha \ln \frac{h}{u^t},$$

with first integral

$$\frac{h}{u^t} = \mathcal{E}.$$

where $\mathcal{E}$ is a constant

($\mathcal{E}$ is the injection energy per unit baryon mass needed to bring baryons at infinity to the same internal state as that in the star, lower them, give them the speed of the baryons in a fluid element, open a space to put them, and inject them into the star).

COMPACT BINARIES: QUASISTATIONARY EQUILIBRIA

In the Newtonian limit, because a binary system does not radiate, it is stationary in a rotating frame.
Because radiation appears only in the 2 1/2 post-Newtonian order --

to order $(v/c)^5 \times$ Newtonian theory,

one computes radiation for most of the inspiral from a stationary post-Newtonian orbit.

Time translations in a rotating frame are generated by a helical Killing vector k^α.

For irrotational flow, a good approximation at late stages of inspiral, u^α is not along a Killing vector, but there is still a first integral:

We can always write the velocity u^α in the 3+1 form

$$u^\alpha = u^t \left(k^\alpha + \mathrm{v}^\alpha \right)$$

with $\mathrm{v}^\alpha n_\alpha = 0$. For irrotational flow, $h\mathrm{v}_\alpha = \nabla_\alpha \Psi$
and the first integral of the relativistic Euler equation is

$$\frac{h}{u^t} + h u_\alpha \mathrm{v}^\alpha = \mathcal{E}$$

In the Newtonian limit and in the curved spacetime of a rotating star, k^α has the form

$$k^\alpha = t^\alpha + \Omega\phi^\alpha.$$

where t^α and ϕ^α are timelike and rotational Killing vectors. For a stationary binary system in GR, one can choose t and ϕ coordinates for which k^α has this form with $t^\alpha = \partial_t$ and $\phi^\alpha = \partial_\phi$.

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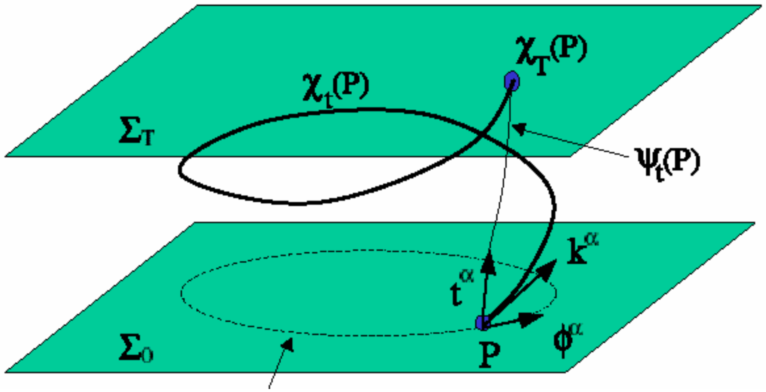
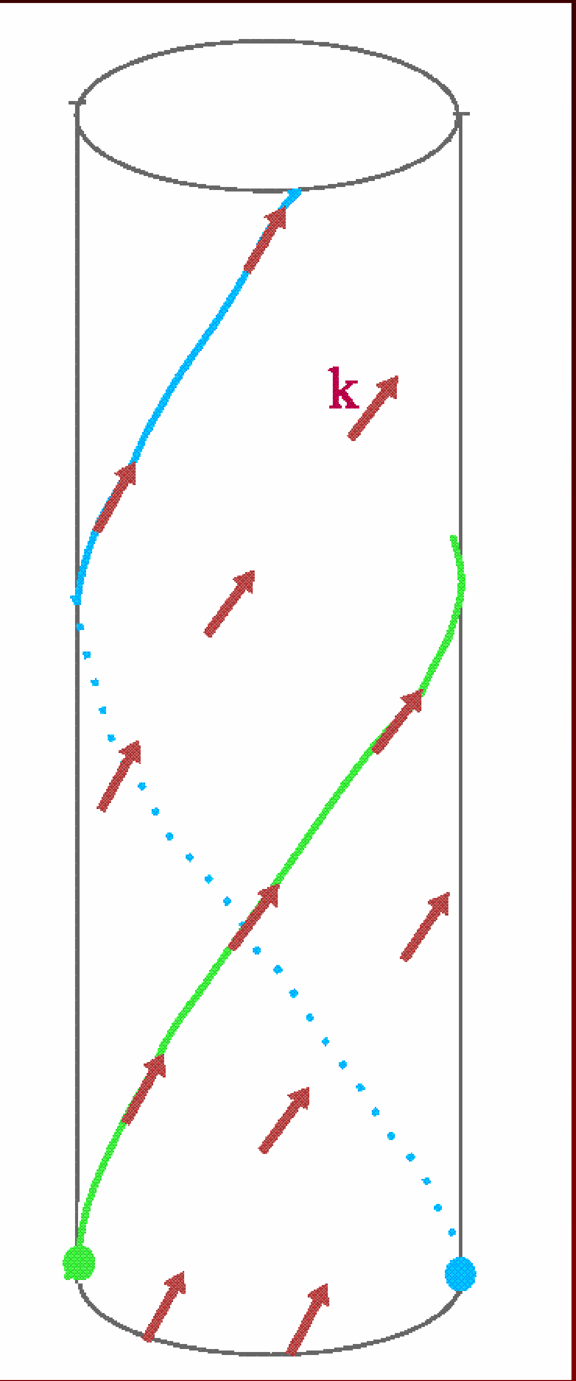
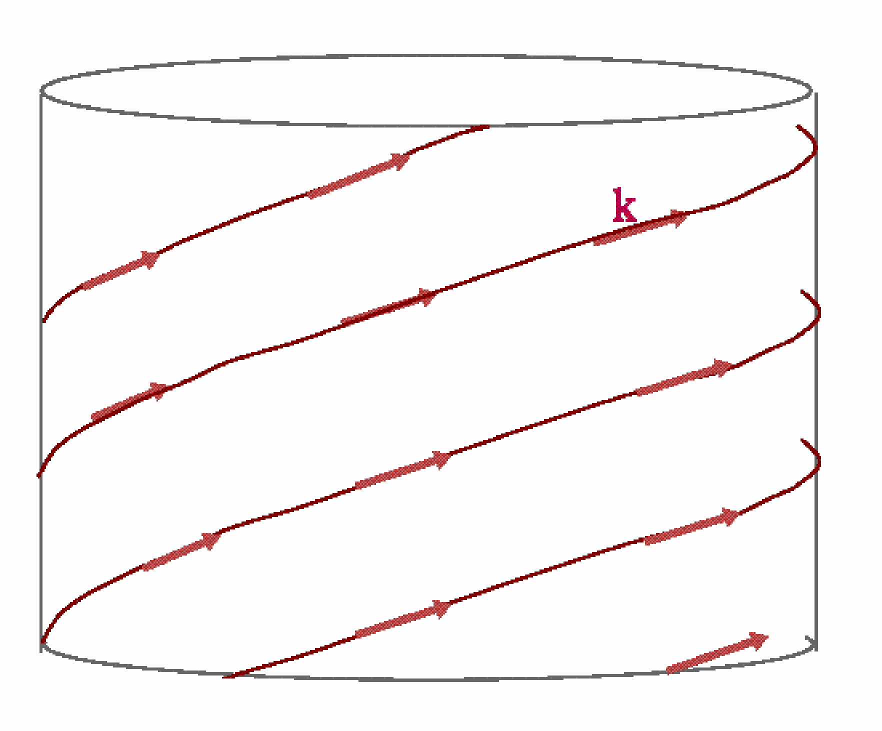
(One can define a helical KV by its helical structure in spacetime: There is a unique period T for which each point P is timelike-separated from the corresponding point a parameter distance T later long the orbit.)





k^{α} is timelike near the fluid

k^{α} is spacelike outside the *light cylinder* at $\varpi\,\Omega = 1$



Although k^α is spacelike outside a large cylinder, one can, as usual, introduce a 3+1 split associated with a spacelike hypersurface Σ. Evolution along k^α can again be expressed in terms of a lapse and shift,

$$k^\alpha = \alpha\, n^\alpha + \beta^\alpha$$

n^α is the future pointing unit normal to Σ

β^α a vector on Σ

In a chart t, x^i, for which Σ is a t = constant surface, the metric is

$$ds^2 = (-\alpha^2 + \beta_j\beta^j)dt^2 + 2\beta_j dx^j dt + \gamma_{ij} dx^i dx^j,$$

where α, β^a, and γ_{ab} are the lapse function, shift vector, and 3-D spatial metric of Σ.

Quasiequilibrium models are based on helically symmetric spacetimes in which
a set of field equations and
the equation of hydrostatic equilibrium,
are solved for the independent
metric potentials and
the fluid density.

From, e.g., the angular velocity and multipole moments of a model, one can compute the energy radiated and construct a quasiequilibrium sequence.

Until recently, these sequences (and initial data for NS binaries) were restricted to spatially conformally flat metrics, the IWM (Isenberg-Wilson-Mathews) approximation.

ISENBERG-WILSON-MATHEWS ANSATZ: SPATIALLY CONFORMALLY FLAT METRIC

$\gamma_{ab} = \psi^4 f_{ab}, \quad f_{ab}$ flat.

Five metric potentials ψ, α, β^{a} are found from five components of the Einstein equation:

1 Hamiltonian constraint

$$\Delta\psi = -2\pi\psi^5\rho_H - \frac{\psi^5}{8}K_a{}^b K_b{}^a,$$

3 components of the Momentum constraint

$$\partial_b(\sqrt{\gamma}K_a{}^b) = 8\pi j_a\sqrt{\gamma},$$

Spatial trace of Einstein eq: $\gamma^{\alpha\beta}(G_{\alpha\beta}-8\pi\, T_{\alpha\beta})$, with $\dot{K} = 0$

$$\Delta(\alpha\psi) = 2\pi(\alpha\psi)\psi^4(\rho_H + 2S_c{}^c) + \frac{7}{8}(\alpha\psi)\psi^4 K_a{}^b K_b{}^a$$

IWM solutions have 5, not 6, metric functions and satisfy only 5 of the 6 independent components of the Einstein equation. An IWM spacetime agrees with an exact solution only to 1st post-Newtonian order.

Initial data then has some spurious radiation and cannot accurately enforce the $\Omega(r)$ relation.
Orbits from the data can be elliptical. One improves the data by adding the asymptotic equality between Komar and ADM mass.

To do better, we need the remaining metric degree of freedom.

DATA IN WAVELESS APPROXIMATION

Initial value equations are satisfied, and
time derivatives are artificially dropped in remaining field equations to replace hyperbolic equations for the tracefree part of the spatial metric by elliptic equations.

(Shibata, Uryu, Friedman 2004,
Bonazzola, Gourgoulhon, Grandclement, Novak 2003,
Schafer, Gopkumar 2003,
Uryu, Limousin, Shibata, JF 05)

Accurate to 2PN, with accuracy to 3PN
(ignoring radiation at 2 ½ PN) possible

Metric and extrinsic curvature

$\tilde{\gamma}_{ab}$: the conformal metric defined by $\gamma_{ab} = \psi^4 \tilde{\gamma}_{ab}$.

A_{ab} : a trace free part of K_{ab}, $A_{ab} = \psi^{-4} \tilde{A}_{ab} := K_{ab} - \frac{1}{3}\gamma_{ab} K$.

f_{ab} : the flat metric $h_{ab} = \tilde{\gamma}_{ab} - f_{ab}$:

Formulation

- Einstein equation $G_{\alpha\beta} = 8\pi T_{\alpha\beta}$ decomposed as follows.

Constraints : $(G_{\alpha\beta} - 8\pi T_{\alpha\beta})n^\alpha n^\beta = 0, \quad (G_{\alpha\beta} - 8\pi T_{\alpha\beta})\gamma^\alpha{}_a n^\beta = 0.$

Trace of projection to Σ_t : $(G_{\alpha\beta} - 8\pi T_{\alpha\beta})\gamma^{\alpha\beta} = 0.$

Tr free part of projection to Σ_t : $(G_{\alpha\beta} - 8\pi T_{\alpha\beta})(\gamma^\alpha{}_a \gamma^\beta{}_b - \frac{1}{3}\gamma_{ab}\gamma^{\alpha\beta}) = 0.$

Impose stationarity conditions:

Fluid: $\mathcal{L}_k\left(\rho u^t\sqrt{-g}\right)=0=\gamma_a{}^{\alpha}\mathcal{L}_k\left(hu_{\alpha}\right)$

Metric $\mathcal{L}_k K_{ab}=0,$ $\partial_t\gamma_{ab}=0$

Then the tracefree G_{ab} equation

$$(G_{\alpha\beta}-8\pi T_{\alpha\beta})(\gamma^{\alpha}{}_{a}\gamma^{\beta}{}_{b}-\tfrac{1}{3}\gamma_{ab}\gamma^{\alpha\beta})=0.$$

becomes elliptic.

Separate out flat Laplacian Δh_{ab}

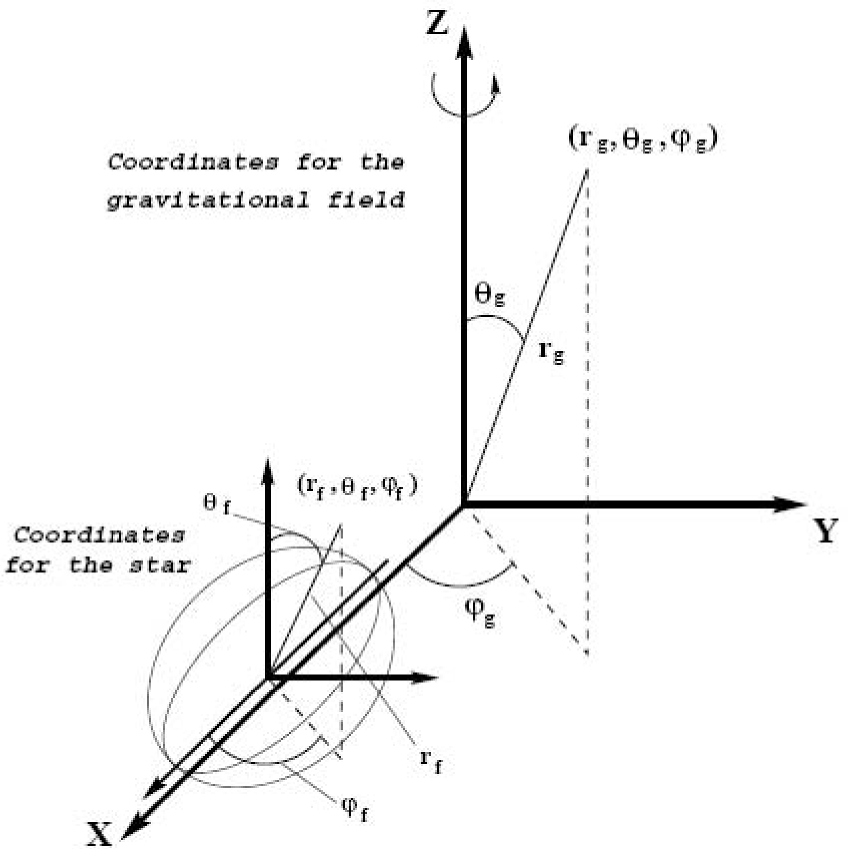
to solve elliptic eqn iteratively for h_{ab}

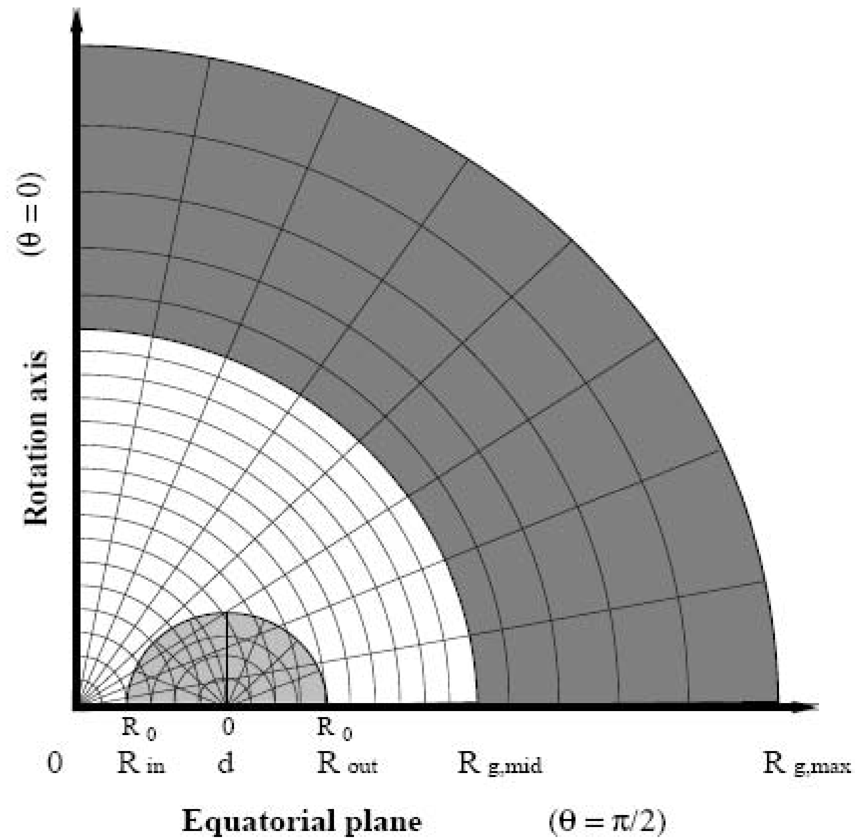
$$\mathring{\Delta} h_{ab} = 2\left(\mathcal{E}_{ab} - \frac{1}{3}\tilde{\gamma}_{ab}\tilde{\gamma}^{cd}\mathcal{E}_{cd}\right) - \frac{1}{3}\tilde{\gamma}_{ab}\mathring{D}^{e} h^{cd}\mathring{D}_{e} h_{cd}$$

where

$$\mathcal{E}_{ab} := \frac{1}{\alpha}(\Delta h_{ab} + \tilde{A}_{ab}) + R^{\mathrm{NL}}_{ab} + {}^{3}\tilde{R}^{\psi}_{ab} - 2\psi^{4}\tilde{A}_{ac}\tilde{A}_{b}{}^{c} - \frac{1}{\alpha}D_a D_b \alpha - 8\pi S_{ab}$$

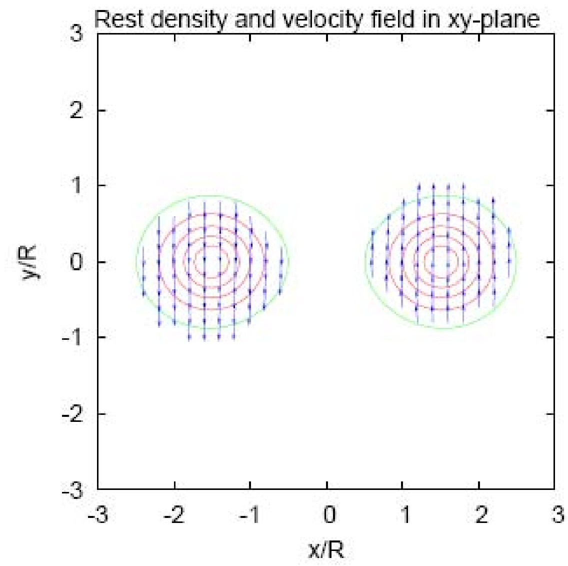
○ Coordinates and region for numerical computation.





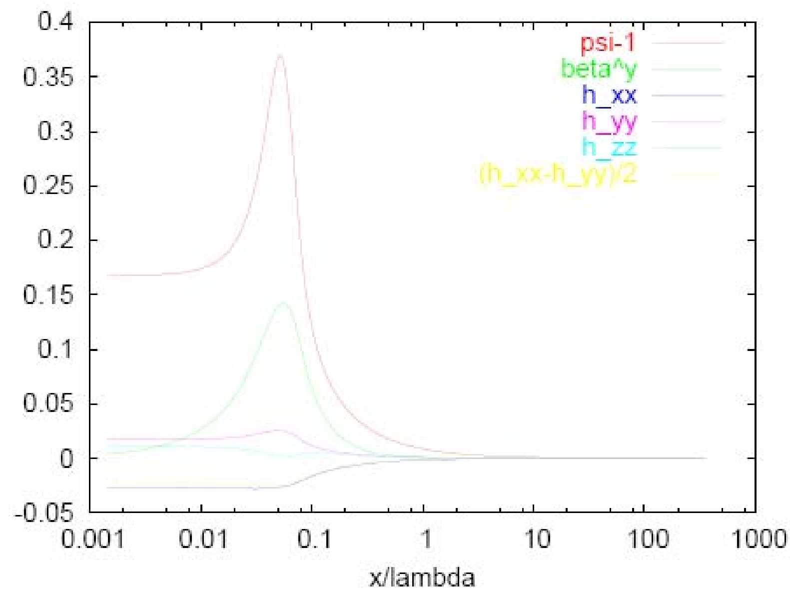
Numerical solution

(Uryu, Shibata, Limousin, JF)



n=1 polytrope,

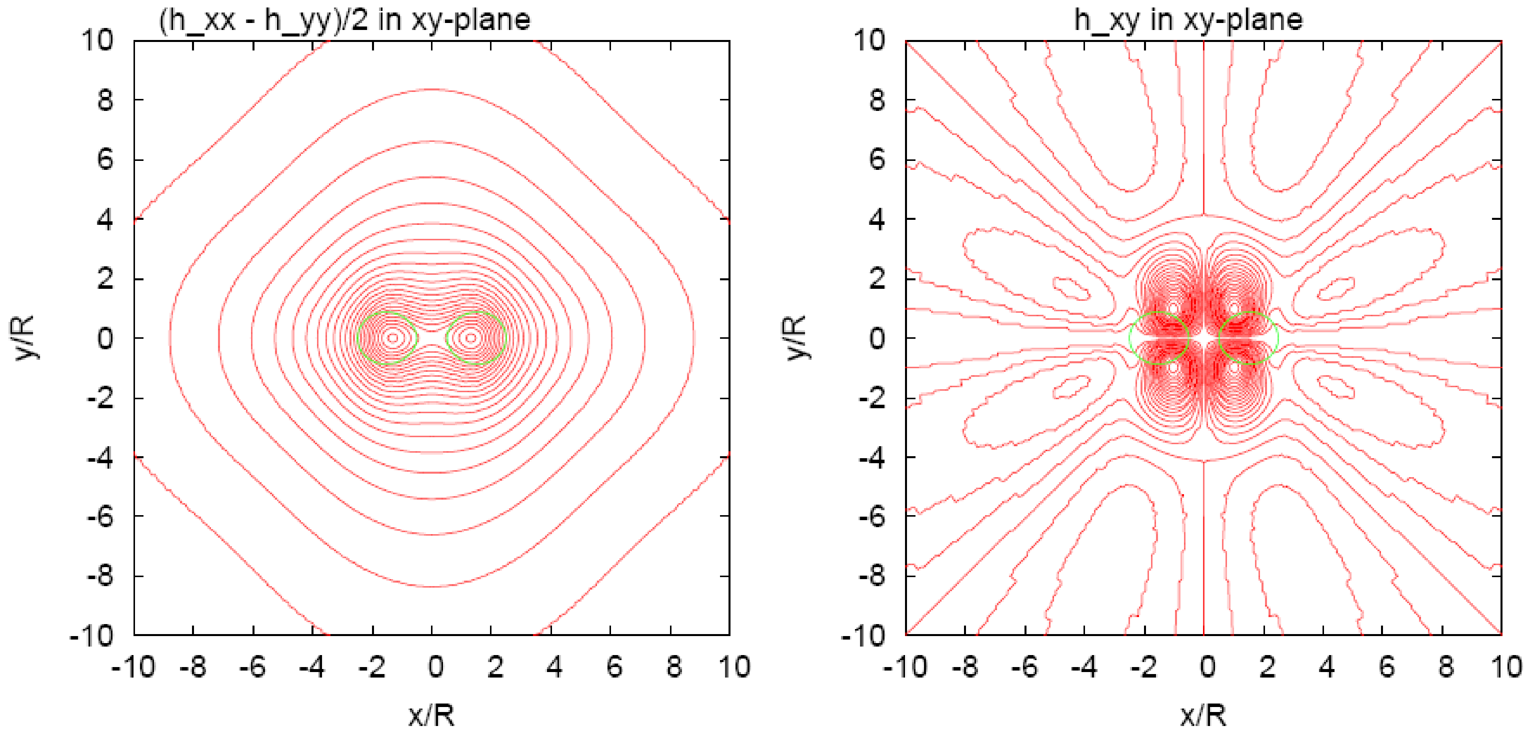
M/R = 0.2, d/R = 1.5



Non-conformally flat parts of

$$h_{ab} := \gamma_{ab} - f_{ab}$$

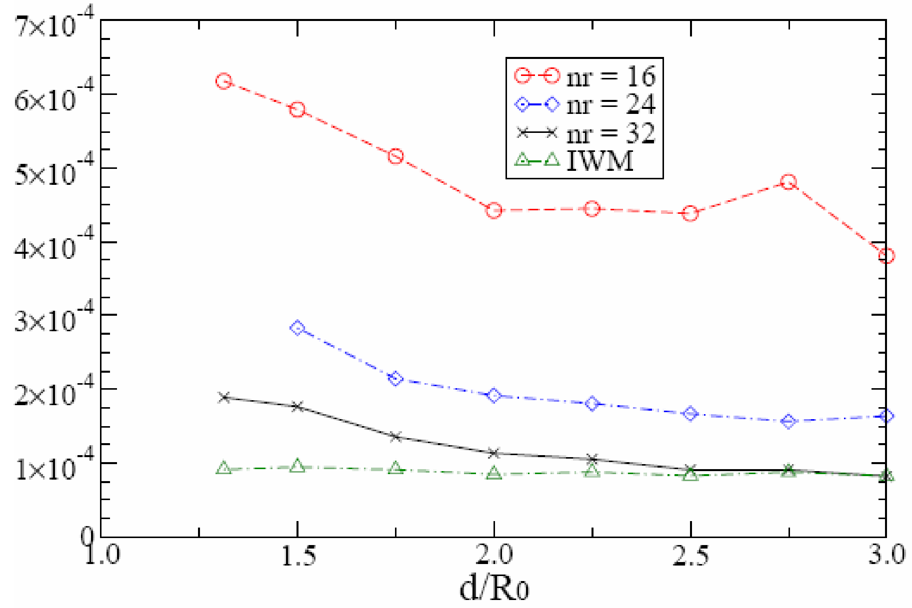
Contour lines of h_{ab}



- Beig and Ashtekar-Magnon showed that that $M_K = M_{ADM}$ for stationary solutions. By modifying Beig's proof, we obtain sufficient asymptotic conditions for the equality. (Shibata, Uryu, JF 04) It is, in particular, satisfied in the present waveless formalism and serves as a check on the accuracy of the solution: $(M_K - M_{ADM})/M_{ADM} < 0.02\%$

$$M_{\mathrm{ADM}} = -\frac{1}{2\pi}\int_\infty D^a\psi\, dS_a$$

$$M_{\mathrm{K}} = \frac{1}{4\pi}\int_\infty D^a\alpha\, dS_a$$



Helically symmetric solutions

Related work on helically symmetric exact solutions for BH's, NS's, and toy models by

Blackburn, Detweiler

Detweiler,

Whelan, Krivan, Price

Whelan, Beetle, Krivan, Price

JF, Uryu, Shibata

Andrade, Beetle, Blinov, Burko, Bromley, Cranor, Owen, Price

Torre

Klein

Bromley, Owen, Price

JF, Uryu

Two advantanges of helically symmetric solutions to the full set of Einstein equations:

- Energy in gravitational radiation is controlled (smaller than that of the outgoing solution, for practical grids).

- By satisfying a full set of Einstein-Euler equations, one enforces a circular orbit.

Because data obtained by solving the initial value equations alone or from an (spatially conformally flat) approximation satisfy a truncated set of field equations, they yield elliptical orbits.

An exact helically symmmetric solution is

not asymptotically flat,

because the energy radiated at all past times is present on a spacelike hypersurface.

At a distance of a few wavelengths (larger than the present grid size) the energy is dominated by the mass of the binary system, and the solution appears to be asymptotically flat.

Only at distances larger than about 10^4 M is the energy in the radiation field comparable to the mass of the binary system.

Work in progress by
Uryu, Price, Beetle, Bromley;
Shin'ichirou Yoshida, JF, Jocelyn Read,
Ben Bromley, Koji Uryu
seeks to solve the full equations without truncation for a helically symmetric spacetime.

In a helically symmetric spacetime, the constraints remain elliptic, but the dynamical equations have a mixed elliptic-hyperbolic character:
elliptic where k^α is spacelike and hyperbolic where k^α is timelike:

$$g^{\alpha\beta}\nabla_\alpha\nabla_\beta\psi = \text{Lower derivative terms} \equiv S.$$

Flat space wave equation with helical symmetry:

$$k^{\alpha}\nabla_{\alpha}\psi = 0 \;\Rightarrow\; \partial_t\psi = \Omega\,\partial_{\phi}\psi$$

$$\begin{aligned}\Box\psi &= \left[\left(-\partial_t{}^2 + \frac{1}{\varpi^2}\partial_{\phi}{}^2\right) + \nabla_r{}^2 + \frac{1}{r^2}\nabla_{\theta}{}^2\right]\psi \\ &= \left[\left(-\Omega^2 + \frac{1}{\varpi^2}\right)\partial_{\phi}{}^2 + \nabla_r{}^2 + \frac{1}{r^2}\nabla_{\theta}{}^2\right]\psi\end{aligned}$$

In general,

$$\pounds_k\Psi = 0 \;\Rightarrow\; g^{\alpha\beta}\nabla_{\alpha}\nabla_{\beta}\Psi = j^{ab}\nabla_a\nabla_b\Psi$$

j^{ab} Euclidean, k^{α} timelike,

j^{ab} Lorentzian, k^{α} spacelike.

Problem is not intrinsically elliptic, BUT after spherical harmonic decomposition, have coupled system of elliptic equations (Helmholtz eqs), each of form

$$(\nabla^2 + m^2\Omega^2)\psi_{lm} = S_{lm}(\psi)Y_{lm}$$

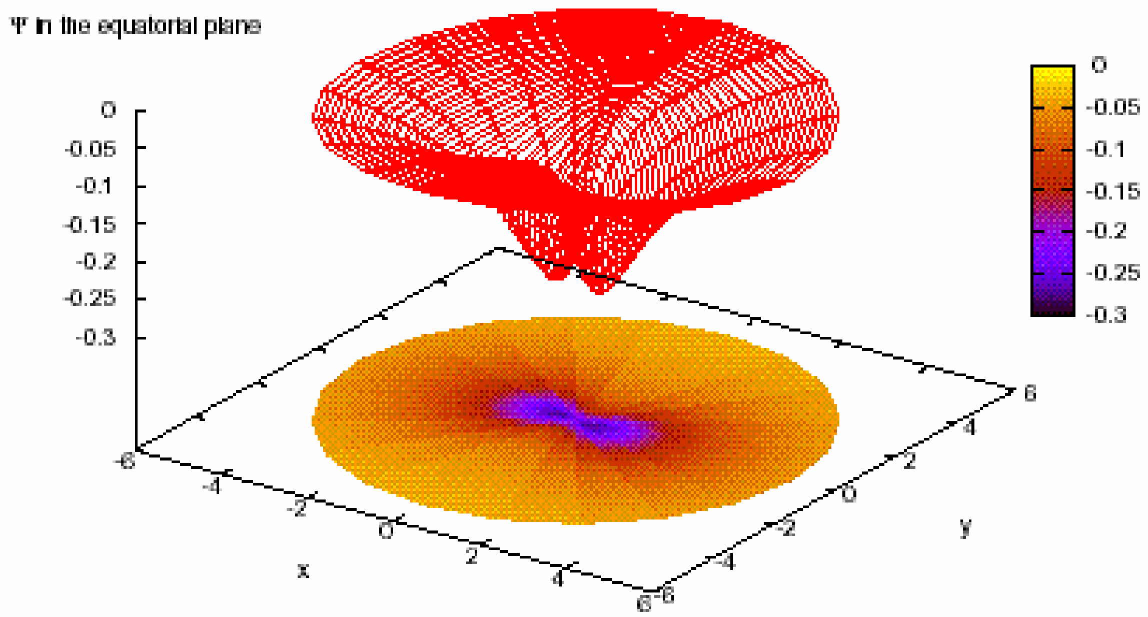
Toy problems:

Nonlinear wave equation and two orbiting point scalar charges as the source :

$$\Box\psi = -\lambda\psi^3 + S \qquad \Box\psi = -\lambda(\nabla\psi)^2 + S$$

$$\Box\psi = -\lambda\psi\Box\psi + S$$

Remarkably, for each case, iteration converges for λ >>1, when sign of λ opposite to sign of S. (Yoshida,Bromley,Read,Uryu,JF).



But for sign of λ same as sign of S, convergence is limited to λ of order unity.

Range of convergence is roughly independent of iteration method used:

Newton-Raphson (Bromley)

Successive inversion of $\Box \overset{n+1}{\psi} = \lambda N(\overset{n}{\psi})$ (Yoshida, Read)

Successive inversion of $\Delta \overset{n+1}{\psi} = \lambda N(\overset{n}{\psi}) + \Omega^2 \partial_\phi{}^2 \psi$ (Uryu,Yoshida)

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(Yoshida, Read)

Successive inversion of $\Delta \overset{n+1}{\psi} = \lambda N(\overset{n}{\psi}) + \Omega^2 \partial_\phi{}^2 \psi$
(Uryu,Yoshida)

But extend range of λ by placing boundary closer to the source.

Boundary	Minimum lambda for convergence
2.5	-6
5	-4
10	-3
15	-2.5
30	-2.3
60	-2.2

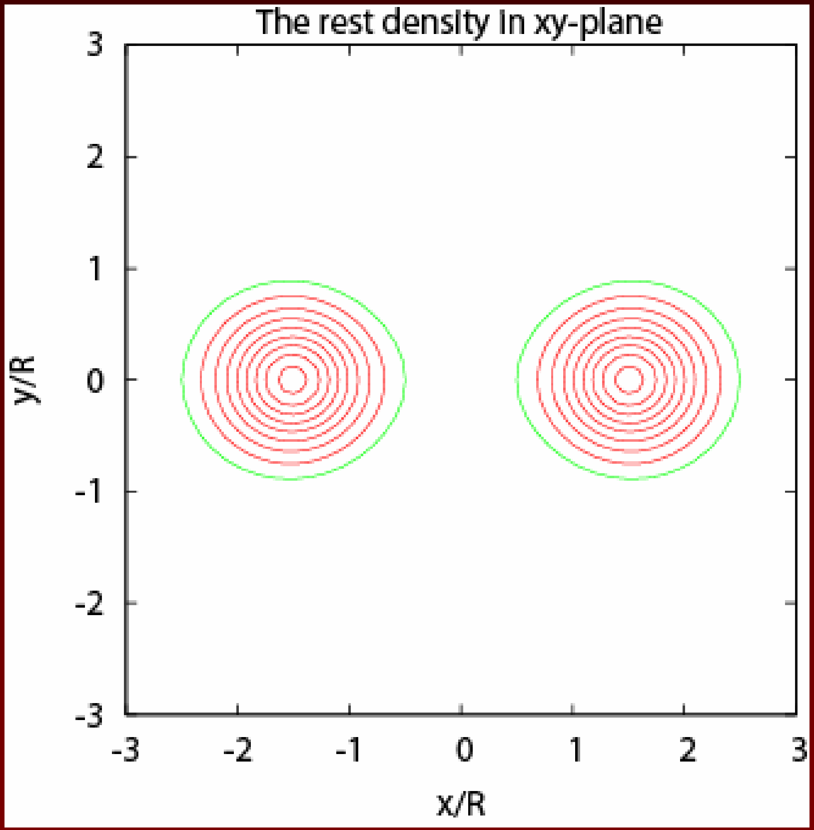
RECENT SUCCESS (Uryu):

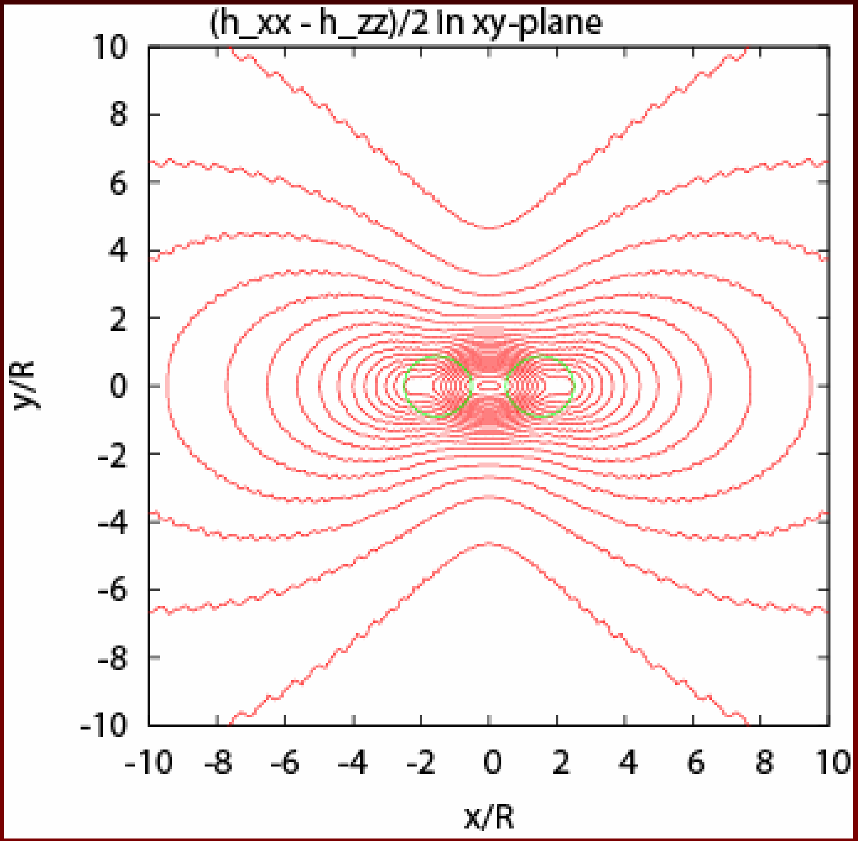
USE WAVELESS TO SET B.C. FOR HELICALLY SYMMETRIC SOLUTION :

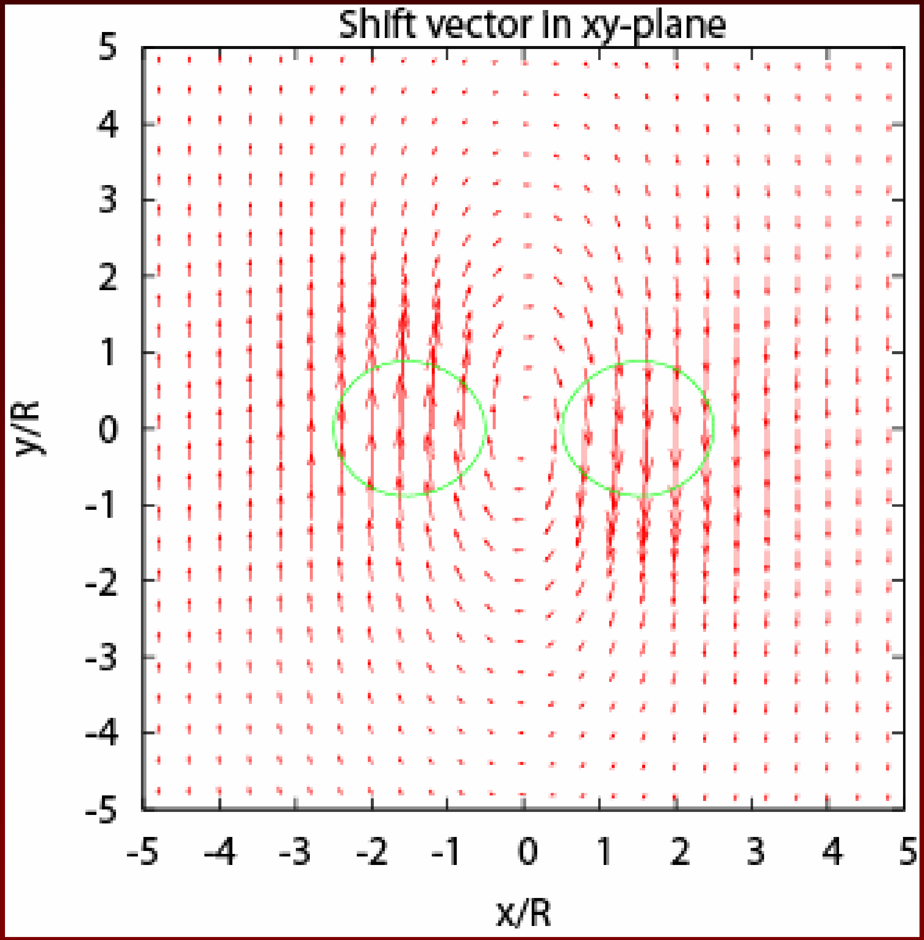
WAVELESS OUTSIDE $r = \pi/\Omega$

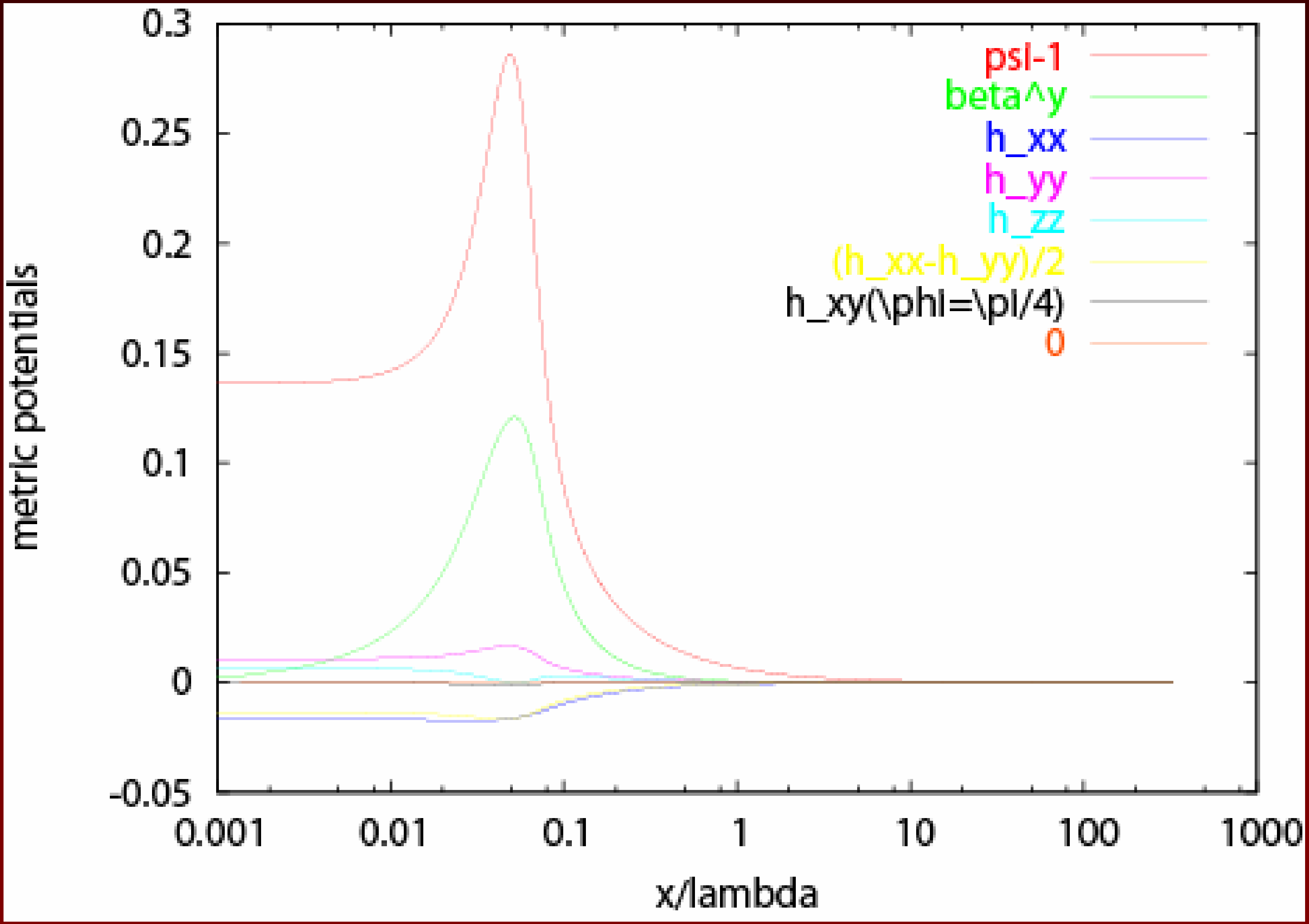
HELICALLY SYMMETRIC INSIDE

THEN THE FULL BINARY NEUTRON-STAR CODE CONVERGES!


The rest density in xy-plane
y/R
x/R
3
2
1
0
-1
-2
-3

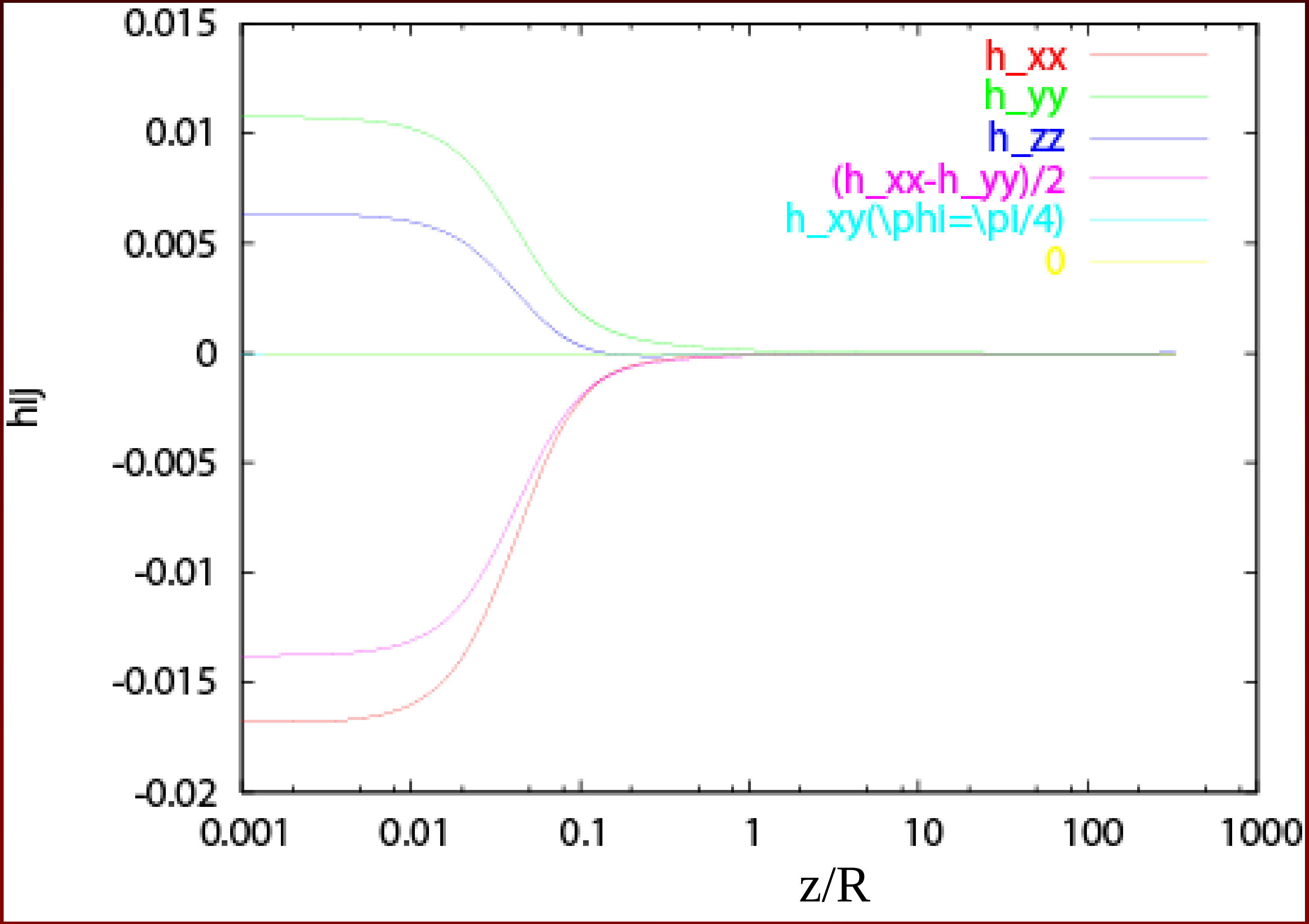

(h_xx - h_zz)/2 in xy-plane
y/R
x/R
10
8
6
4
2
0
-2
-4
-6
-8
-10


Shift vector in xy-plane
y/R
x/R
5
4
3
2
1
0
-1
-2
-3
-4
-5

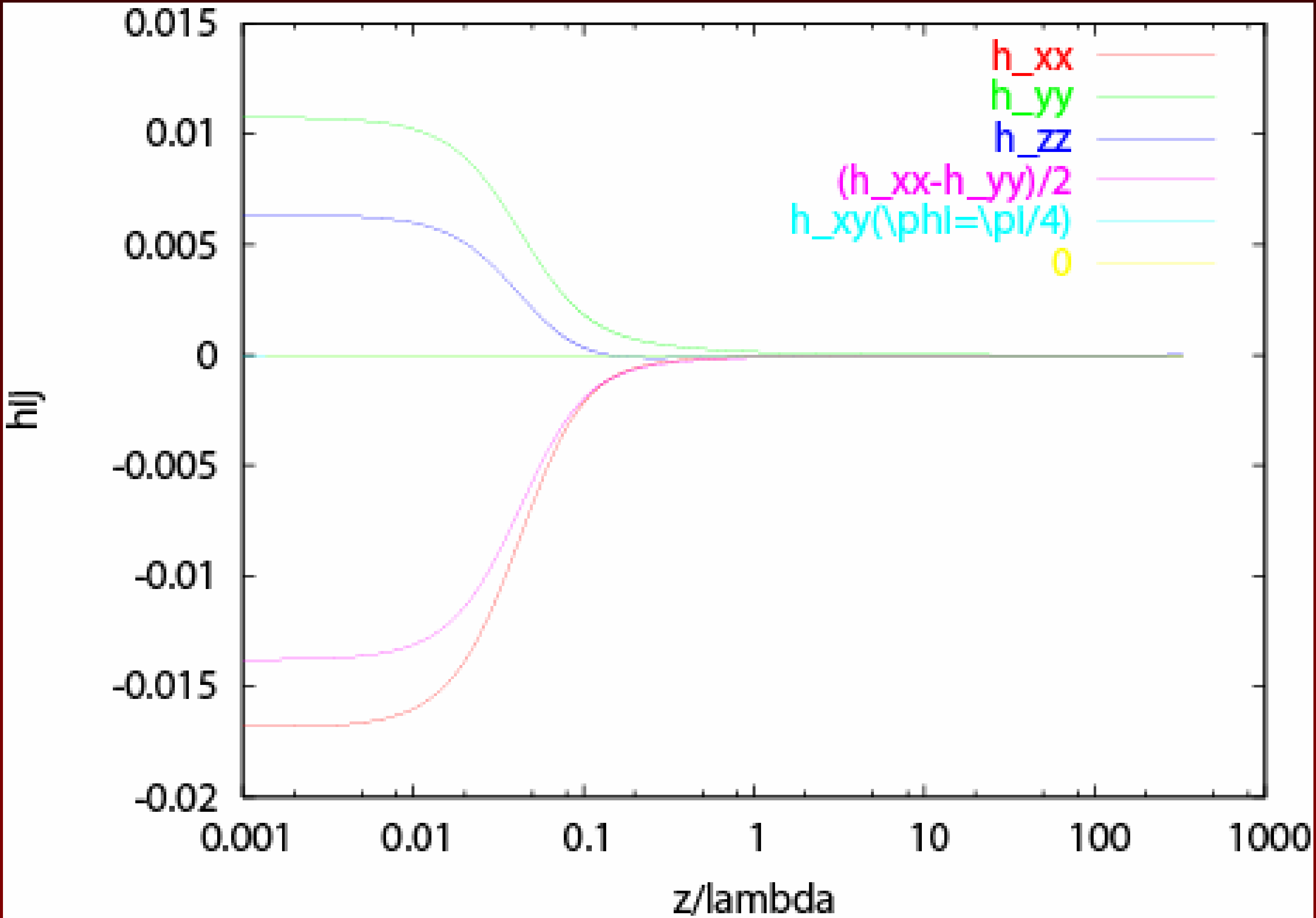


metric potentials
x/lambda
psi-1
beta^y
h_xx
h_yy
h_zz
(h_xx-h_yy)/2
h_xy(\phi=\pi/4)
0
0.3
0.25
0.2
0.15
0.1
0.05
0
-0.05
0.001
0.01
0.1
1
10
100
1000

HELICALLY SYMMETRY INSIDE R = π/Ω



WAVELESS EVERYWHERE



Some recent results from binary NS merger simulations

(Shibata, Taniguchi, Uryu)

Simulations using parametric EOS:

Realistic EOS (T=0) + Ideal gas (thermal)

$p = p_{cold} + p_{thermal}$

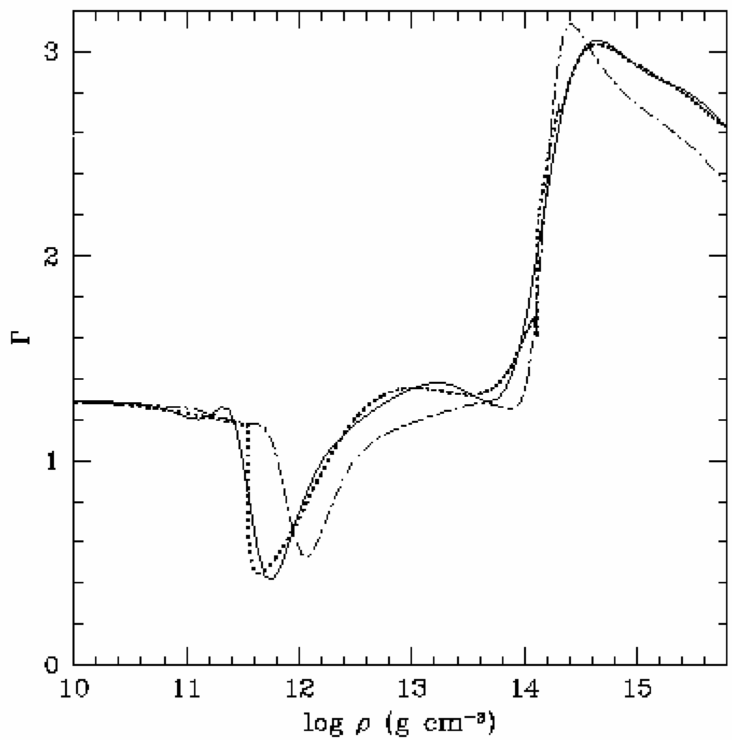
p_{cold} : FPS or SLy

$p_{thermal} = (\Gamma_{th} - 1)\,\rho\varepsilon_{thermal}$: $\varepsilon_{thermal} = \varepsilon - \varepsilon_{cold}$

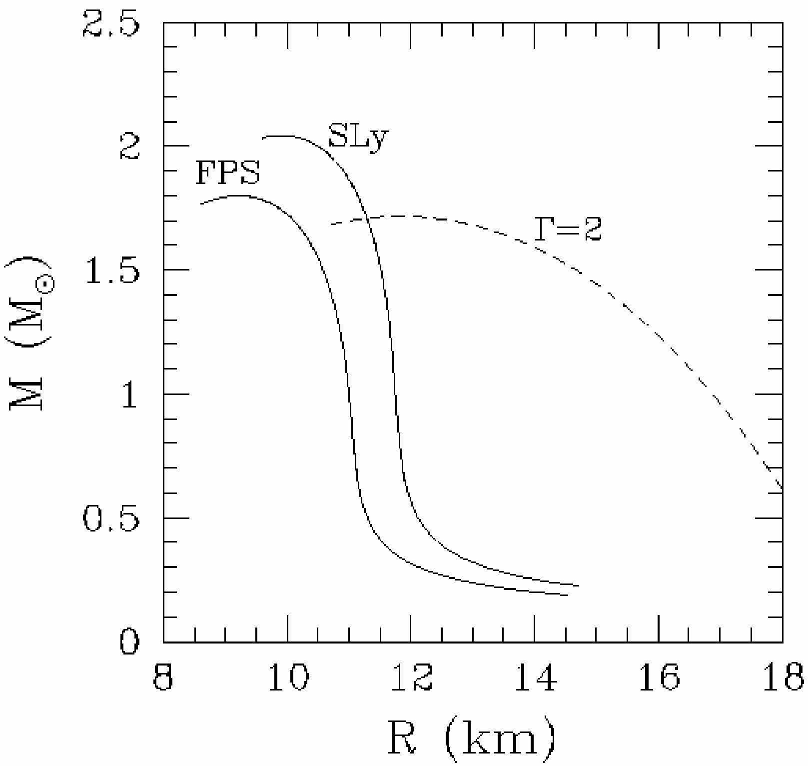
$\Gamma_{th} = 2$, (also 1.3, 1.65).

Cold part of the EOS.

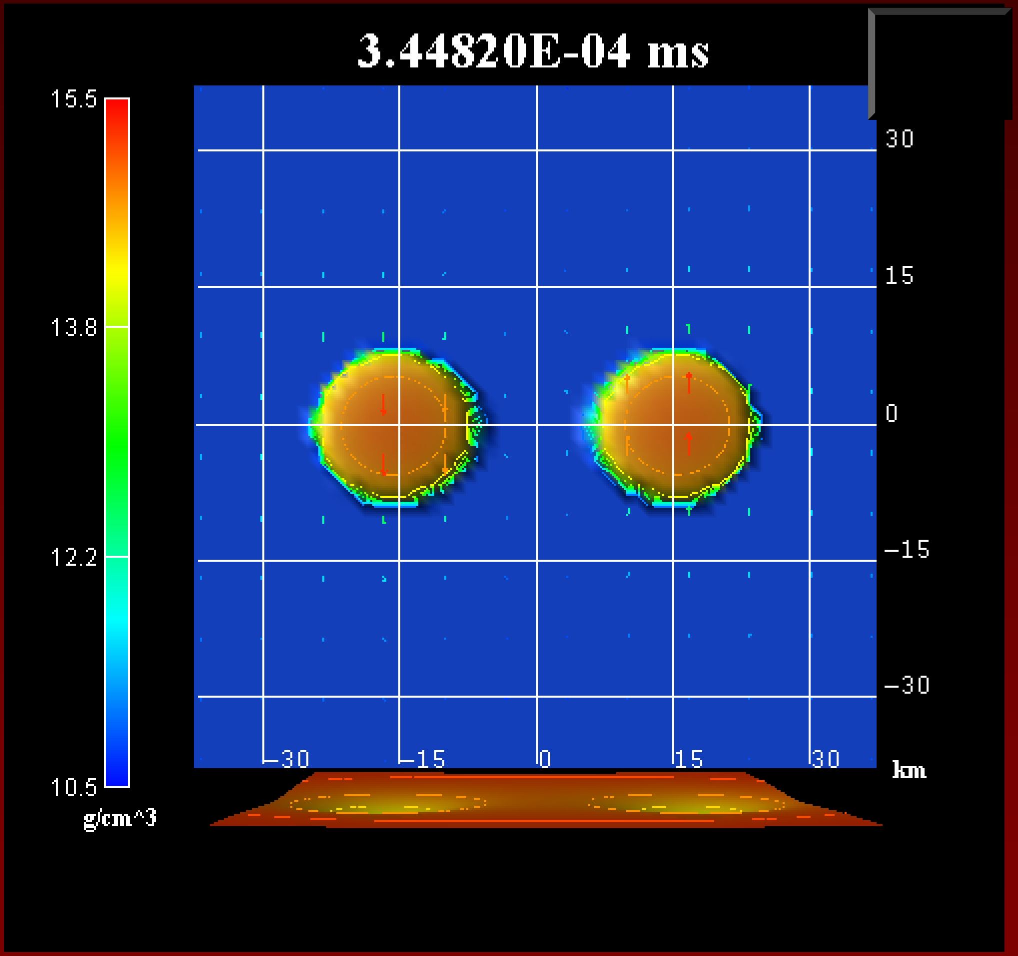
- Stiffer on high density side, compared to earlier simulations $\Gamma=2$ to 2.25.



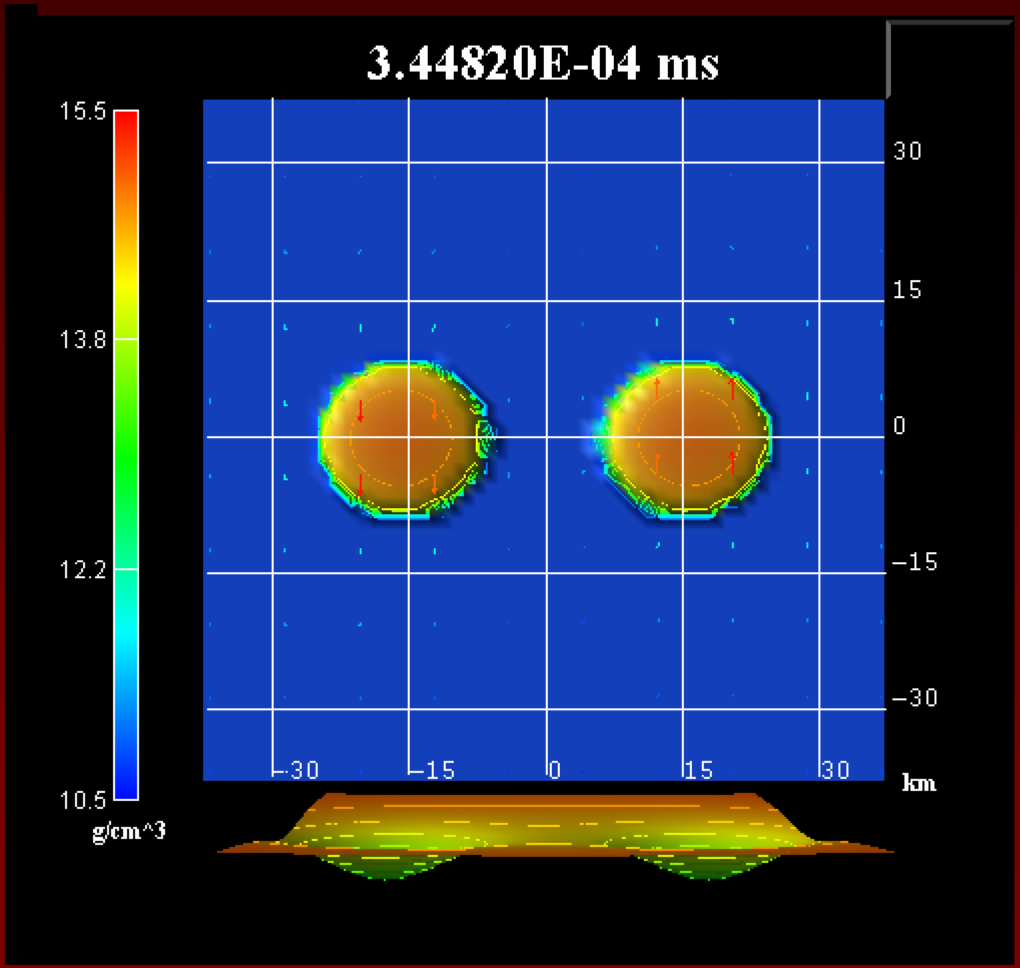
Haensel & Potekhin (2004)



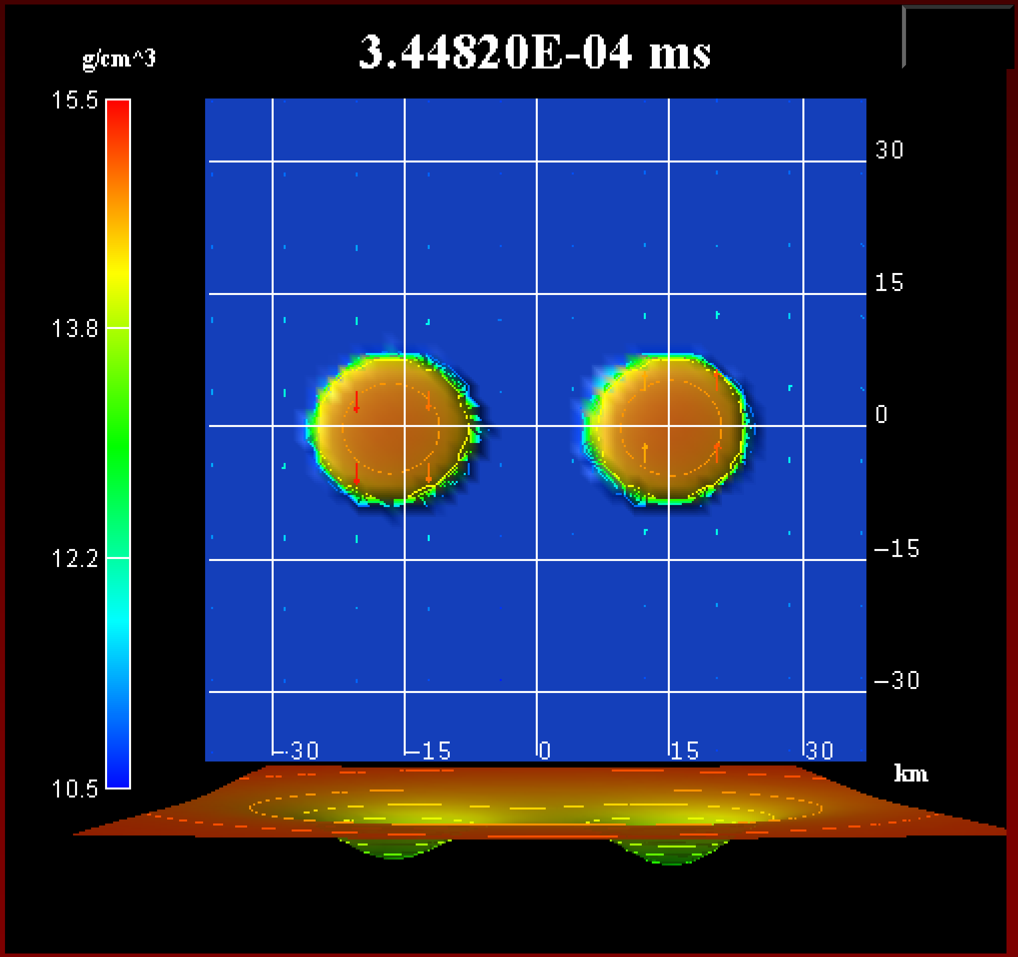
SLy EOS, $1.4\,M_\odot - 1.4\,M_\odot$ merger



SLy EOS, $1.3M_{\odot}-1.3M_{\odot}$ merger.



SLy EOS, $1.35\,M_\odot - 1.25\,M_\odot$ merger.



Results of simulations

- Merger outcome:
  prompt BH, or
  nonaxisymmetric rotating hypermassive NS.

- Mass threshold for prompt BH formation :
  $> 2.7\ M_\odot$ for SLy, $> 2.5\ M_\odot$ for FPS
- Dynamics of merger does not depend on the the mass ratio for $M_1/M_2 = 0.9$ to1.

- J/M^2 of BH is 0.7-0.8, whence oscillation frequency > 6.5 - 7 kHz.

Implications

- Reliable templates for late inspiral, joining the PN waveform are close.
- Once we find gravitational waves from binary neutron stars, it will be important to look for the last part of the train:
  GW from oscillations at 3-4 kHz.
- Observing the frequencies of quasinormal modes would constrain the EOS.

Results of simulations

- Merger products: prompt BH, or nonaxisymmetric rotating hypermassive NS.
  (In earlier $\Gamma=2$ results, hypermassive NS were nearly axisymmetric.)
- Mass threshold for prompt BH formation :
  $> 2.7\ M_\odot$ for SLy, $> 2.5\ M_\odot$ for FPS
- Dynamics of merger does not depend on the the mass ratio for $M_1/M_2 = 0.9$ to1.
- When a BH forms, the disk mass depends weakly on the mass ratio, $M_1/M_2 = 0.9$ to 1, $M_{disk} < 1\%$.
- J/M^2 of BH is 0.7-0.8, whence QNM frequency > 6.5 - 7 kHz.

Hypermassive NS as a strong GW and neutrino emitter.

- A nonaxisymmetric rotating hypermassive NS is a strong emitter of quasi-periodic GW around 3 - 4 kHz.
- Effective GW amplitude will be

$h_f = 1.8\times10^{-21}$, $(dE/df/10^{51}\ \mathrm{erg/Hz})^{1/2}$, 100Mpc/r .)

- Hypermassive NS collapses to BH in <100 ms, because GW emission carries J away.
  Neutrino cooling may not govern the collapse, because its emission timescale is 1 to 10 s.
  (Outer region of hypermassive NS is 10 to 20MeV, neutrino energy flux » 10^{53}.)

Summary

- We are approaching reliable templates for binary merger, joining the PN waveform to the merger waveform.
- Once we find gravitational waves from binary neutron stars, it will be important to look for the last part of the train:
  GW from oscillations at 3-4 kHz
- Observing the frequencies of quasinormal modes constrains the EOS.